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Neighbor vector centrality of complex networks based on neighbors degree distribution

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Abstract. We introduce a novel centrality metric, the neighbor vector centrality. It is a measurement of node importance with respect to the degree distribution of the node neighbors. This centrality is explored in the context of several networks. We use attack vulnerability simulation to compared our approach with three standard centrality approaches. While for real-world network our method outperforms the other three metrics, for synthetic networks it shows a slightly weak properties but still a good measure overall. There is no significant correlation of our method with network size, average degree or assortativity. In summary, neighbor vector centrality presents a novel measurement of node importance, which has a better performance to reduce dynamics of real-world complex networks.

1 Introduction

Complex networks can represent various complex systems, where the main elements of the system are considered as nodes and interaction between them are presented as edges. In recent years, we have witnessed an intense research activity on networks by the scientific community. Many of them are excited and greatly extend our understanding of real-world systems. These include the synchronization transition [1], epidemic spreading [2], community structures [3], topological hierarchy [4] and transmission of information [5], etc. Among them, the concept of centrality has been discussed for more than 60 years. Many of the algorithms originated in the field of social network analysis [6]. These metrics are typically used to measure how important nodes are in the topology, which is responsible for critical features of real-world networks, such as robustness against failures/attacks [7], and the absence of a threshold for percolation [8] and even ranking of websites [9]. The most commonly used centrality measures are: degree, betweenness, closeness and eigenvector centrality [10].

Degree centrality measures importance as those nodes have the highest number of edges. Betweenness centrality measures importance from a flow perspective by examining the number of shortest paths passing through the node of interest [11]. Eigenvector centrality calculates a type of relative importance, i.e., important nodes must be neighbors of other important nodes [12], whereas closeness centrality can be defined as the total graph distance of a node from all the others [13]. On the other hand, attack tolerance of complex networks is always an interesting topic scientists concern [7,14]. Intentionally removing nodes or edges, the impact of certain attack strategy can be explored. If removals by the descending order of different centrality metrics are used respectively, the damage they cause is an indicator that how well the measures work in evaluating node importance.

In this paper, it is our objective to develop a novel centrality measure. We use attack simulations to test the proposed neighbor vector centrality. Based on degree centrality, it is more efficient to evaluate node importance and without the computation complexity like betweenness. By using an iterated logarithm, we unfold a degree value into a vector representing the number of neighbors with different degree levels. This approach makes node importance more distinguished. In several real-world networks, attack strategies based on deleting nodes by the descending order are used to test our method and three traditional ones. The result shows that our method can easily display the neighbors degree distribution of a node and measure its importance. In most of the networks, the strategy based on neighbor vector centrality has the best performance. In the tests of breaking down networks, deteriorating transmission efficiency and decreasing controllability of networks, neighbor vector centrality scores the most points.

The rest of this paper is organized as follows. In Section 2, the definition of neighbor vector centrality is given, and we also discuss what the definition means and how to compare nodes by it. In Section 3, to prove the validity and generality of the method, attack simulations on several real-world networks are presented. Based on the test results, we compare our approach with other different

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three centrality metrics. In Section 4, the conclusion is given.

2 Definition of neighbor vector centrality

There are three major reasons we want to devise a novel metric, (1) to evaluate node importance by its neighbors importance and distribution characteristic; (2) to avoid potential computational complexity; and (3) to find a better way to measure topological significance of nodes.

Experiment results show degree centrality is a preferable option in many cases since it is easy to compute and high efficient to evaluate node importance. However, it lacks the consideration of neighbors importance, which is critical in many real-life cases. For example, even if I have more friends on Facebook than Barack Obama, it does not mean I am more important than him. By contrast, betweenness and other similar metrics in the family are relatively difficult to compute, but they appropriately consider the global information of topology. Therefore, to achieve our goals, we present a method to unfold a single degree value into a vector representing the number of neighbors with different degree levels. We devise our approach based on degree centrality to inherit its computational simplicity. With different neighbors divided into different classes, the distinction of neighbors is emphasized. In general, evaluation of the difference of node neighbors and a better measurement of node importance are our focus.

In the following paragraphs of this section, we exploit the iterated logarithm function (Def. 1) and unfold a degree value into a vector. The definition of neighbor vector centrality is given by Definition 2. After a description of the novel centrality measure, the method how nodes can be compared with others by neighbor vector centrality is discussed in Definition 3. We also give a proof that such a comparison makes the nodes a strict total order set.

Definition 1 *(iterated logarithm). In computer science, the iterated logarithm of n, written as* $\log^* n$*, is the number of times the logarithm function must be interatively applied before the result is less than or equal to 1:*

$$\log^* n := \begin{cases} 0 & \text{if} \quad n \leq 1; \\ 1 + \log^*(\log n) & \text{if} \quad n > 1. \end{cases} \tag{1}$$

In many real-world complex network, such as the Internet, degree zero and one are usually the less important nodes, but always with a large number (due to the well-known power-law distribution of degree). It does make sense to present the number of nodes with degree zero and one. Consequently, we use a base-2 iterated logarithm. Under such condition, the value range of this function is given in Table 1.

Definition 2 *(neighbor vector centrality). For any node* v_i *in node set* V*, we have its neighbors set* N_i *representing all the nodes connected to* v_i*. And number of nodes in* N_i *is* n_i*. Assuming there is a six-dimensional vector* x_i *for node* v_i*,* $x_i = (x_{i0}, x_{i1}, x_{i2}, x_{i3}, x_{i4}, x_{i5})^T$*. Each* x_{ij} *in vector* x_i *can be calculated by* $x_{ij} = \sum_{k=1}^{n_i} C_j(\log^*(M(v_k)))$*,* $j = 0, 1, 2, 3, 4, 5$*, where* $M(v_k)$ *is a value of given metric, which in our case is degree centrality of node* v_k*,* $v_k \in N_i$*.* $C_j(\xi)$ *is a function given as follows:*

$$C_j(\xi) := \begin{cases} 0, & \text{if} \quad \xi \neq j; \\ 1, & \text{if} \quad \xi = j. \end{cases} \tag{2}$$

Denote $C_d(v_i) = \sum_{j=0}^{5} x_{ij}$*. Thus,* x_i *is the neighbor vector of node* v_i*. Let* $m_i = \|x_i\|_2 = \sqrt{\sum_{j=0}^{5}(x_{ij})^2}$*, where* $\|\cdot\|_2$ *is the 2-norm of a vector. And the number of non-zero elements of neighbor vector* x_i *can be calculated by the following equation:*

$$f_{nz}(x_{ij}) : = \begin{cases} 0, & \text{if} \quad x_{ij} = 0; \\ 1, & \text{if} \quad x_{ij} \neq 0. \end{cases}$$

$$F(x_i) = \sum_{j=0}^{5} f_{nz}(x_{ij}). \tag{3}$$

By this definition, we have another vector $s_i = (F(x_i), m_i)^{\mathrm{T}}$, and for a node set V there is a corresponding vector set $S = \{s_1, \ldots s_i, \ldots s_n\}$. The importance of nodes is decided by vector s_i according to Definition 3.

The basic idea of neighbor vector centrality is that we like to unfold a single degree value into a vector, which makes nodes more distinguishable. For each element of the vector, a number indicates how many neighbors a node has with a certain degree level. Namely, for a node v_i, x_{i0} in x_i is the sum of 1-degree and 0-degree neighbors, x_{i1} is the number of 2-degree neighbors, x_{i2} presents the sum of 3-degree and 4-degree neighbors, x_{i3} is the number of neighbors with degree more than 4 but less than 17, x_{i4} contains the number of neighbors with degree between 17 and 65 536, and any neighbor of v_i has degree larger than 65 536 is counted by x_{i5}. Therefore, the neighbors of v_i are considered as several classes with different degree levels. Looking at a neighbor vector x_i of some node v_i, immediately we know how many low-degree nodes x_i connected to (x_{i0} and x_{i1}, etc.), and how many high-degree neighbors it has (x_{i4} and x_{i5}).

Additionally, this partition is chosen because we want to see the difference of degree distribution as clear as

Table 1. Value range of base-2 iterated logarithm by Definition 1.

x	$\lg^*(x)$
$(-\infty, 1]$	0
$(1, 2]$	1
$(2, 4]$	2
$(4, 16]$	3
$(16, 65536]$	4
$(65536, 2^{65536}]$	5

possible and integer range is preferable for analysis convenience. However, it can be adjusted by using another base for the iterated logarithm function in Definition 1, such as any real number greater than 1.

Definition 3. *Given two vectors $s_q \in S$ and $s_p \in S$, and denote $\xi = s_q - s_p = (\xi_f, \xi_m)^T$. Let $\xi_* = \{\xi_f | \xi_f \neq 0\} \bigcup \{\xi_m | \xi_f = 0\}$. A binary relation $\prec$ is defined as follows. If $\xi_* < 0$, then $s_q \prec s_p$, correspondingly the neighbor vector centrality of vector x_q is smaller then x_p. Similarly, if $\xi_* > 0$, then $s_p \prec s_q$, the neighbor vector x_q is larger then x_p. When $\xi_* = 0$, x_q and x_p are equal, denoted as $x_q = x_p$.*

Moreover, we need to prove that our approach can be used to sort all the nodes of target network. Hence, the proof is given as follows to ensure the strict total order of the set S.

Theorem 1. *The above definition of binary relation $\prec$ over the set S is a strict total order on the set S, i.e., the binary relation $\prec$ satisfies:*

(1) *$\forall x,\ y \in S$, exactly one of $x \prec y$, $y \prec x$ and $x = y$ is true (trichotomy);*
(2) *$\forall x,\ y \in S$, if $x \prec y$, then it is not the case that $y \prec x$ (asymmetry);*
(3) *$\forall x,\ y,\ z \in S$, if $x \prec y$ and $y \prec z$, then $x \prec z$ (transitivity).*

The proof is given as follows.

First, for any $x, y \in S$, let $\xi = x - y$. It is true that exactly one of $\xi_* < 0$, $\xi_* > 0$ and $\xi_* = 0$ holds. Thus, from the definition of binary relation $\prec$, the binary relation $\prec$ is trichotomous.

Secondly, if $x \prec y$, then $\xi_* < 0$. Consequently, it is not the case that $\xi_* > 0$ and hence $y \prec x$ is not true.

Finally, for any $x, y, z \in S$, let $\alpha = x - y$, $\beta = y - z$, and $\xi = \alpha + \beta$. Then, $x - z = (x - y) + (y - z) = \alpha + \beta = \xi$. By the definition of binary relation $\prec$, if $x \prec y$, then either $\alpha_f < 0$ or $\alpha_f = 0$, $\alpha_m < 0$. Similarly, $y \prec z$ implies that either $\beta_f < 0$ or $\beta_f = 0$, $\beta_m < 0$. One can deduce that either $\xi_f = \alpha_f + \beta_f < 0$ or $\xi_f = 0$, $\xi_m = \alpha_m + \beta_m < 0$. Thus, we have $\xi_* = \{\xi_f | \xi_f \neq 0\} \bigcup \{\xi_m | \xi_f = 0\} < 0$ and hence $x \prec z$. Therefore, the binary relation $\prec$ is transitive.

Based on this proof, we know our definition of neighbor vector and binary relation $\prec$ make the node set a strict total order. Thus, it can be sorted to weigh node importance.

Another issue need to be addressed is an ideal centrality metric can be used to distinguish the difference between every pair of nodes while evaluating node importance. It would be the best if every node has a different value of the metric. Based on this idea, we define repeat rate and discrimination rate as follows.

Definition 4. *For a centrality measurement, if the metric value for a node is identical with any other node, it is a repeated metric value. Theoretically, the two nodes cannot be distinguished from each other. In a n-node network, repeat rate of a given metric can be defined as $RR = \frac{r}{n}$, where r is the number of repeated metric values. Likewise, discrimination rate of the metric is $DR = \frac{n-r}{n}$.*

Table 2. Value range of discrimination rate of four centrality measurements in nine complex networks. The DR of proposed method is not the highest, but it is close to other metrics and much better than degree centrality.

Centrality type	Max(DR)	Min(DR)	Average(DR)
Our method	52.94%	0.15%	20.36%
Degree	54.35%	0.00%	9.66%
Closeness	60.00%	0.02%	28.09%
Betweenness	99.99%	1.30%	46.22%
Eigenvector	100.00%	5.7%	50.96%

If the discrimination rate (DR) of a centrality measurement is higher, nodes in the network are more distinguishable. In Table 2, it shows that DR of five centrality measures in the 9 networks used in this paper. We can see average DR of degree centrality is the smallest (9.66%), and proposed neighbor vector increases DR to 20.36%, which is very close to other standard metrics.

3 Attack vulnerability of complex networks

To measure the effectiveness of neighbor vector centrality, simulations of attack vulnerability are presented in this section. Degree centrality (DC), betweenness centrality (BC), eigenvector centrality (EC) and neighbor vector centrality (VC) are compared in the simulation. The experiment is designed as follows. (1) We compute all centrality measurements of the initial networks; (2) then select the highest top-n nodes in the network by a metric; (3) we intentionally remove them and their edges; (4) and evaluate the impact of such an attack on the network. (5) Gradually, we increase n from 0.1% to 20% and repeat the process.

To prove the validity and generality of the proposed method, this section includes six real-life directed networks and three directed synthetic networks. The synthetic networks includes a generated *Barabási-Albert* (BA) model [15], a small-world (SW) model [16] and an *Erdös-Rényi* (ER) [17] random graph. We use the neural network of Elegans [16] to test biological networks. For social networks, we use Zachary's karate club [18], coauthors in network science [19], network of characters in the novel Les Miserables [20] and a consulting network [21]. A data set representing Internet IPv6 topology is also added to test large-scale physical network. The details of those networks are in Table 3.

For BA model, an edge is established by selecting a source node and then choosing a preferred high-degree node (preferential attachment). In this paper, the direction of the edge is set as from the source to the preferential node. In the ER and SW models, the direction is decided in a similar fashion, i.e., the edge points to the subsequent selected node. On the other hand, the edge direction in the coauthor network is decided by the author order in papers.

Table 3. Basic information of networks used in this paper. nD is controllability measurement (explained in Sect. 3.3), $\langle k \rangle$ is average degree and r is assortativity.

Network	Nodes	Edges	nD	$\langle k \rangle$	r
Coauthors	1589	2742	0.402	1.73	0.46
Internet	12746	29119	0.41	2.28	–0.029
Consulting	46	879	0.065	19.10	–0.749
Les Miserables	60	237	0.25	3.95	–0.1258
C. Elegans	306	2345	0.268	7.66	–0.1520
Club	34	78	0.588	2.29	–0.4756
BA	5000	4999	0.664	0.99	–0.0691
SW	2000	8000	0.0165	4	–0.0362
ER	1000	2000	0.409	2	–0.0132

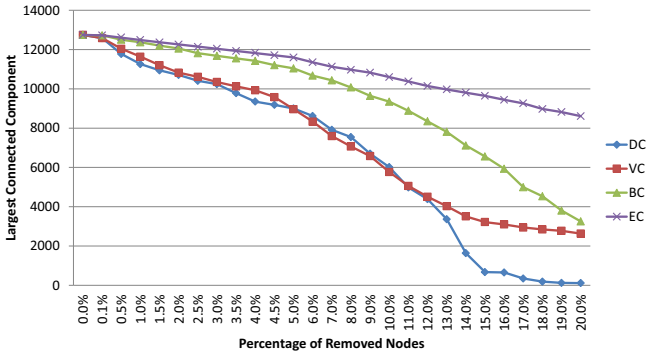

Fig. 1. Largest connected component during attack simulation on Internet. Red square line indicates the strategy based on neighbor vector.

And the routing order determines the direction of edge in the Internet.

3.1 Network connectivity

During the attack simulation, networks are damaged and broken into components. The number of components indicates the degree of such a break-down. The largest connected component is also considered, since it presents the connectivity of the left-over networks. Some of the results are shown in Figures 1 and 2. As we can see, neighbor vector centrality is the best choice in breaking the Internet topology into components, but it is very similar to degree centrality when removed nodes is less than 12%.

Since we only care about which centrality measurement has the best performance during the attack, to quantify the result, one point is assigned to the centrality that performs the best in an attack simulation step. If two centrality measures are tied for the first place, they both get 1 point. Thus, DC gets 20 points in Figure 1 while VC gains 8 points. On the other hand, VC gets 23 points in Figure 2 as DC has 5 points. We will discuss the total scores in the final section. In this section, only scores

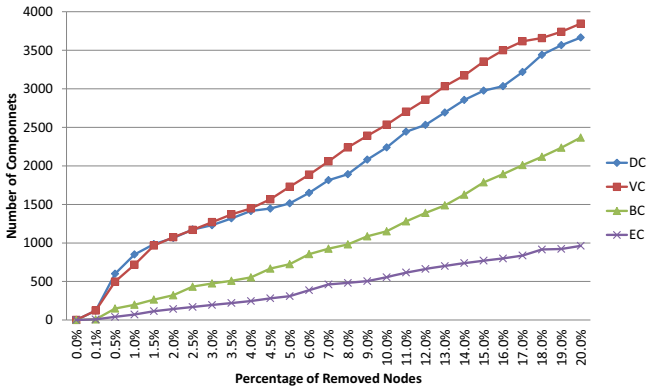

Fig. 2. Total component number during attack simulation on Internet. In most of the time, the proposed VC is better than the other three metrics, since it creates more components.

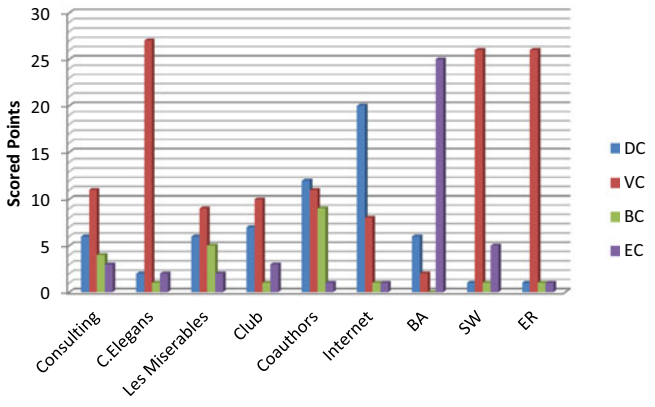

Fig. 3. Points centrality metrics scored from cutting largest connected components (LCC). VC is the best strategy in the attack on first four networks.

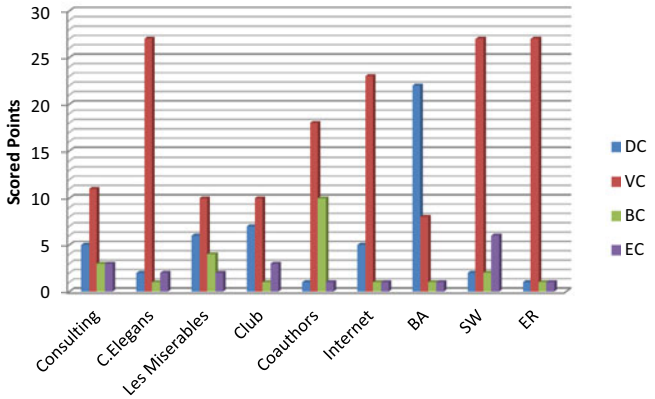

Fig. 4. Points centrality metrics scored from increasing components. VC outperforms the other methods except in BA model.

on the sub-item test are presented. Based on this idea, points scored by the different centrality metrics in destroying largest connected components and increasing the number of components are shown in Figures 3 and 4, respectively. For the Internet, the result is consistent with Figures 1 and 2.

According to our observation, neighbor vector centrality is not the best choice to reduce the connectivity of BA

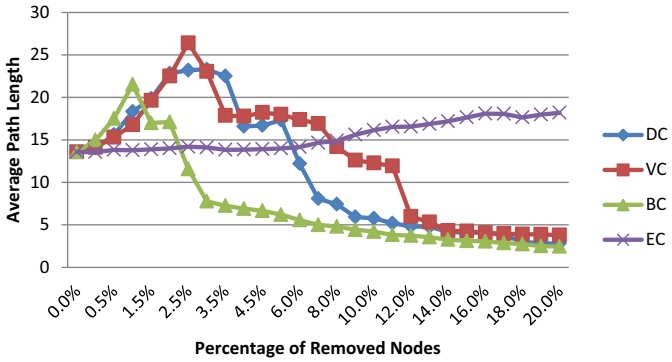

Fig. 5. Average path length l of Internet during the attack simulation. Since EC fails to break down Internet, VC increases l the largest for most of the time.

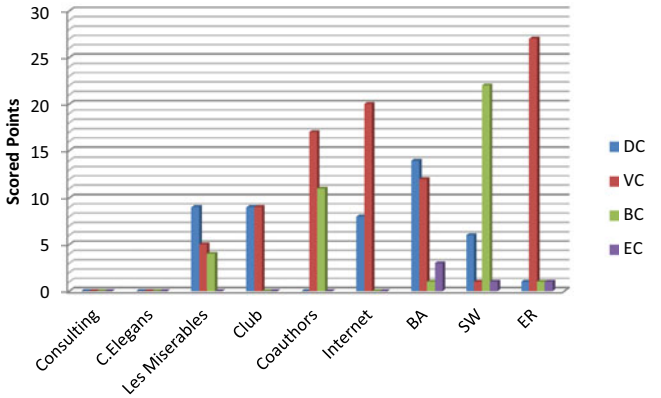

Fig. 6. Points centrality measures scored from reducing transmission efficiency.

model network. DC or EC is a better approach. Meanwhile, although DC reduces the LCC of Internet powerfully, VC is still the first option due to a better performance to break up the whole Internet topology. In other newtworks, VC is the second to none in reducing the network connectivity.

3.2 Transmission efficiency

Unusually, average path length l [22] indicates transmission efficiency. The smaller l is, the best transmission efficiency is. Assuming that a node fails if it is disconnected from the largest connected component [7], we like to explore transmission efficiency in the functional part of networks during attack. So we compare l of LCC from different results based on different metrics when they are comparable (i.e. LCCs have similar size). For example, in Figure 5, DC and VC are comparable since LCCs of them have similar size (shown in Fig. 1). And the LCCs of EC and BC are much larger. On this basis, DC gets 8 points and VC gets 20 points in reducing the transmission efficiency of Internet. We have the score list in Figure 6.

Apparently, we need a further explanation for Figure 6. As we previously stated, it is rational to compare centrality methods which break down network to the same level.

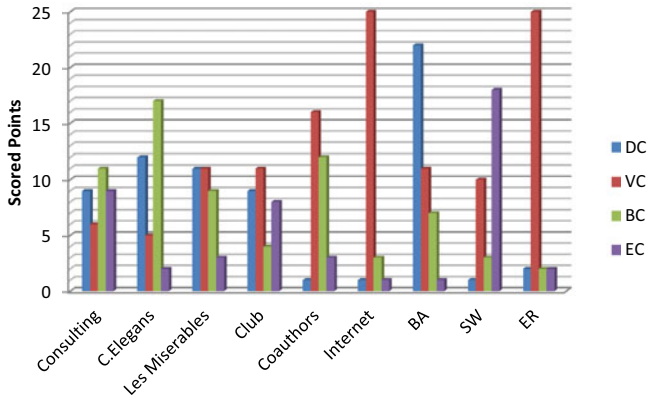

Fig. 7. Points centrality measures scored from nD.

For the consulting network and the C. Elegans network, the attack method based on VC outperforms others in reducing the network connectivity. There is no comparable metrics in the two networks so that all centrality measures score 0 point.

In summary, the proposed VC is quite a great method in reduce the transmission efficiency of networks. In the BA network, it has similar performance like DC. DC is better in Les Miserables, and DC and VC tie up in the Club network.

3.3 Assortativity and controllability

In a recent paper [23], Liu et al. propose a method to measure the controllability of directed complex networks, namely nD. It is a percentage indicating how many nodes are needed to fully control the whole network. The greater nD is, the more nodes are needed. Table 3 is consistent with Liu's results. The Consulting network has the smallest nD and BA model has the largest one. It means the Consulting network only need few input signals (actually three) to fully control the network status, whereas BA model needs independent signals for 62 percents of its nodes to gain the same controllability. During our attack simulation, every network becomes more uncontrollable. The scores of four centrality metrics in disturbing the controllability of the sample networks are shown in Figure 7. The VC preforms well in the simulation. Even in some networks VC fails the first place, such as BA and C. Elegans, we have to mention that the results are quite close (shown in Fig. 8).

On the other hand, Newman devised assortativity [24] to measure the connection tendency of nodes in networks. As Figures 9 and 10 show, there are two types of responds to the attack. Some networks increases assortativity during the attack, such as the Coauthors network, Internet, Consulting network, Les Miserables, the Club network and C. Elegans. Others decrease their assortativity like BA model. We speculate that the reason networks respond in this way is related to their original assortativity value and topological characteristics. Except in the BA model, our approach has a strong tendency to boost the assortativity like in Figure 9. The phenomenon is caused

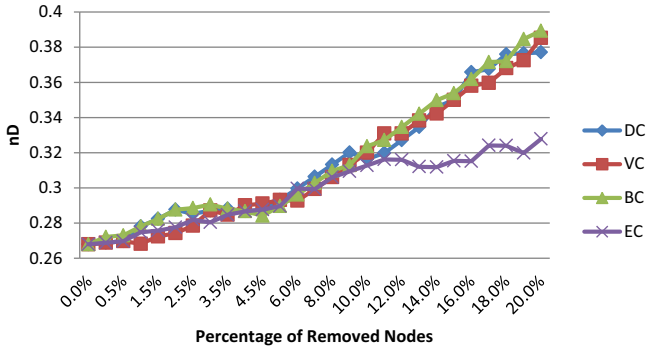

Fig. 8. nD of C. Elegans Network during the attack simulation. DC, BC and VC tie up in most of the time.

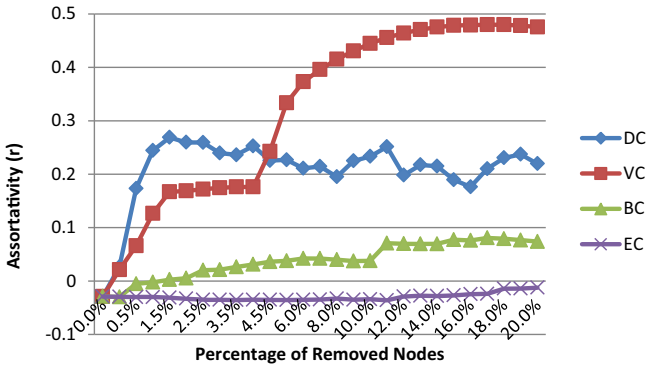

Fig. 9. Assortativity of Internet during the attack simulation. There is a clear increase trend.

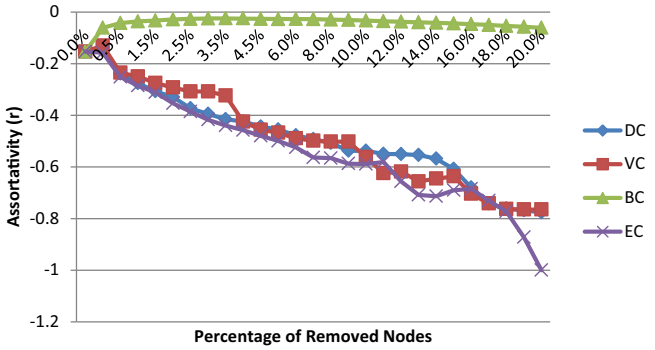

Fig. 10. Assortativity of BA during the attack simulation. There is a clear decrease trend.

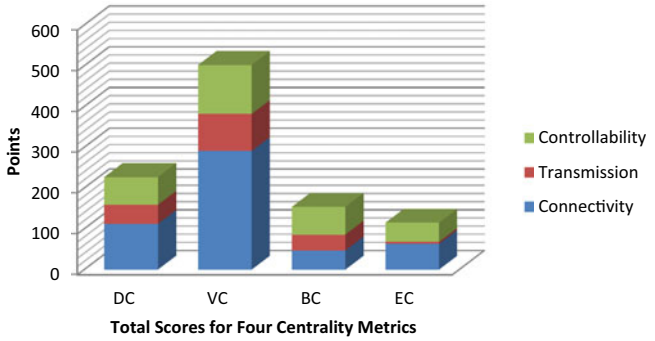

Fig. 11. Final scores for four centrality metrics. Generally, VC has the highest scores and the best performance.

by Definition 3. It requires high-degree node must have a neighbors degree range as wide as possible. Without those nodes, assortativity of network usually increases. However, there is no significant correlation of our method with network size, average degree or assortativity.

Moreover, since assortativity is a characteristic of network, it cannot be evaluated by large or small. We will not calculate the scores the centrality metrics get from it. In Table 3, we can see that C. Elegans network and BA model have nearly the same r indicating their assortativity values are the same. But VC shows totally different performance in the two networks. It infers that our centrality method does not depend on assortativity. Furthermore, the final scores for the four centrality measurements in this network attack game are in Figure 11. It is clear that VC as an enhanced centrality measurement based on degree has the best performance to reduce the dynamics of most complex networks.

When we look back on the effectiveness of VC, one may ask why it can perform better than the others in most of the cases. The key is the consideration of the neighbors difference. We compare nodes by their non-zero elements of VC first, then consider the norm of VC. It guarantees that the node we select must have a wide connection range, some small degree nodes, some medium degree nodes and some bigger ones. When a node with well-connected neighbors is removed as EC usually does, its neighbors are hardly affected due to their great connectivity in the topology. It is the reason EC fails behind in most of the tests. By contrast, when a high-degree node with low-degree neighbors is removed (DC, VC), those low-degree neighbors may have no other optional path to connect to the main topology. After the high-degree node is removed, its low-degree neighbors will be isolated and the whole topology breaks down. That is why DC and VC usually are very competitive. As VC tends to choose nodes with a dispersive neighbors degree distribution, it is efficient to cut connection between high-degree nodes and low-degree nodes in networks. It is the reason VC outraces DC. As to BC, there is no guarantee that the node BC selected is the only way its leaf nodes communicate with other nodes. The nodes are on the shortest paths indeed, but most of the time substitute paths can be found.

4 Conclusion

In this paper, our contribution is the demonstration of a novel centrality metric called neighbor vector centrality. It has an impressive performance to break down real-world complex networks. Comparing with other three traditional centrality measures, it is easy to compute and highly efficient in measuring node importance. Except the BA model network, neighbor vector centrality outperforms others in the simulation. As no obvious dependency on assortativity is observed, our approach does show

merits. Moreover, the method that unfold a single value into a vector is open-end idea. It is also can be used in other metric enhancement, such as betweenness centrality, etc.

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