An Axiomatic Approach to Network Complexity*

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Abstract

Despite the recent wave of interest in the social and physical sciences regarding "complexity," relatively title attention has been given to the logical foundation of complexity measurement. With this in mind, a number of fairly simple, "reasonable" axioms for the measurement of network complexity are here presented, and some of the implications of these axioms are considered. It is shown that the only family of graph complexity measures satisfying the "reasonable" axioms is of limited theoretical utility, and hence that those seeking more interesting measures of complexity must be willing to sacrifice at least one intuitively reasonable constraint. Several existing complexity measures are also described, and are differentiated from one another on an axiomatic basis. Finally, some suggestions are offered regarding future efforts at measuring graph complexity.

Keywords: graph complexity, axiomatization, information content, entropy

1 Introduction

Although the formal theory of complexity extends at least as far back as the development of information theory in the 1950s and 1960s (Shannon, 1948; Rényi, 1961), the past ten years have seen a proliferation of "complexity" related developments in a number of fields (including biology, physics, and computer science). Despite this renewed interest in the notion of complexity, however, little agreement has been reached on the precise meaning of the term, or on how it should

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be applied across contexts. A number of different definitions of complexity have been proposed by researchers in a variety of fields; such definitions vary greatly in their motivations and underlying assumptions, and often differ substantially when applied to the same inputs¹ (see, for instance, Feldman and Crutchfield (1998), Lloyd and Pagels (1988), Li and Vityány (1991), Cover and Thomas (1991), Bennett (1985); a brief review of the best-known complexity measures is contained in Butts (1999)). Given the current confusion about the precise nature of "complexity" per se, there is clearly a great need for work addressing the foundations of complexity measurement.

The context for our investigation into the underpinnings of complexity is that of social network analysis; in particular, our primary interest shall be the quantification of the complexity of digraphs. Since a wide range of social structures can be represented in network form, a treatment of network complexity should permit application to a number of more specific problem domains. Further, social networks offer the added advantage of being well-defined, formal structures on which it should be both possible and desirable to define meaningful complexity measures. Indeed, social and biological networks provide some of the oldest applications of complexity (and related concepts) to the social sciences (see, for instance, Rashevsky (1955), Mowshowitz (1968a; 1968b; 1968c), Freeman (1980; 1984), and Everett (1985)), and an interest in identifying and simplifying "complex" structures has been long-standing within the field (e.g., the work on structural, automorphic and regular equivalence, group definition and identification, etc.). Formal complexity measures already exist for directed graphs (Mowshowitz (1968a; 1968b; 1968c), Freeman (1984), Everett (1985), Butts (1999)), and have been connected to substantively important concepts such as interchangeability of positions (Everett, 1985) and the presence of simplifying structures (Butts, 1999); unfortunately, however, these past treatments have not been derived from a shared formal framework², nor have their foundations been examined.

To attempt, then, to clarify the question of what is (and may be) meant by the "complexity" of a graph, we shall set forth a series of axioms which postulate a set of properties which (it is argued) should be satisfied by reasonable measures of graph complexity. The implications (and consistency) of these axioms will be explored, and a new complexity measure will be introduced which serves as a model for our axioms. Finally, we shall consider a number of existing measures of network complexity, and construct an axiomatic classification which allows one to differentiate between measures on the basis of their general properties; some additional observations regarding the extension of certain of these results to other measures of a given type will be discussed as well.

¹In fact, some measures of complexity are inversely related; see Wolpert and Macready (1998).

²For instance, the measure of Everett (1985) is based on the properties of graph automorphism groups, while that used by Butts (1999) stems from the difficulty in reconstructing graph adjacency matrices using a copy/insertion machine.

2 Initial Axioms

We begin our foundational development by the specification of a set of initial axioms which, we argue, should hold for all measures of graph complexity. It should be noted that the axioms of this section are, to some extent, trivial; nevertheless, they should not be ignored. Even seemingly "trivial" axioms may place important limitations on the family of acceptable measures, limitations which may not be obvious without a thorough accounting of initial assumptions³. Conversely, if we are to begin the process of laying a logical foundation for the measurement of network complexity, it seems sensible to begin with those points on which there is likely to be substantial agreement. Once these initial elements of the framework have been established, we may move to less obvious – and more controversial – axioms.

In the discussion which follows, we shall consider $G = \{V, E\}$ to be a loopless directed graph with vertex set V(G) (with |V(G)| > 0) and arc set $E(G)^4$. Unless specified otherwise, G is assumed to be labeled; however, we may on occasion wish to describe a relabeling of G, L(G), which can be thought of as the directed graph formed by the arcs of E(G) on the relabeling of the elements V(G) as given by L.

The first of our initial axioms serves to define both the input and the output of the complexity measure; specifically:

Axiom 1 (Cardinality) The complexity of a digraph, C(G), must take the set of labeled digraphs into the real numbers.

While the above would seem to be a very reasonable property to consider of a graph complexity measure, it should be noted that one could posit alternatives. Notwithstanding the obvious (but perhaps less interesting⁵) alternatives such as complex or vector-valued complexity measures, it is also the case that one could define graph complexity on a purely ordinal basis. While this might be sensible for an "intuitive" complexity measure, it would be inappropriate for measures based on concrete quantities such as information content, number of automorphism classes, etc. For our purposes, then, it seems reasonable to treat the complexity of a graph as a cardinal real number.

Another property which one might consider as trivial is that of determinism:

Axiom 2 (Determinism) $\forall G, \exists a \in \Re : p(C(G) = a) = 1.$

By Axiom 2 we rule out the possibility of graph complexity being a random variable (except in the most trivial sense). For each labeled graph there must

³For instance, the seemingly "trivial" fact that the geodesic distance between two disconnected nodes is undefined causes measures such as closeness to become undefined on graphs with more than one component; this implication is not always obvious to newcomers to network analysis, and it is certainly not "trivial" in its consequences.

⁴Throughout the text, we shall make these assumptions unless specified otherwise; they should thus be "tacked on" to any statements presented here if not made explicit.

⁵Although, given the lack of consensus over a generally appropriate notion of complexity, the idea of a multidimensional complexity measure has a certain attraction. Such a measure would likely be superfluous, however, as one could always represent its elements individually.

be a unique associated complexity value, which is constant given the complexity measure C.

The above, of course, assumes that C is well-defined for all graphs. This would seem to be an important and reasonable feature, and hence we include among our basic axioms a statement of existence:

Axiom 3 (Existence) C(G) exists \forall finite digraphs G.

Given any finite digraph, then, we expect our complexity measure to take on some value. Note that we leave open the question of infinite-order graphs, as it is not immediately obvious how these should be handled (or whether, in fact, they should be handled at all). As to the nature of the values taken by C(G), we shall further add the restriction of finiteness:

Axiom 4 (Finiteness) \forall finite digraphs G, $\exists a \in \Re : C(G) \leq a$.

Thus, we do not allow C(G) to be unbounded over its domain. Such a restriction is not only intuitive (it is hard to imagine any finite object being infinitely complex), but also useful for practical purposes. Were C(G) to diverge on particular graphs, numerous quantities of interest (such as the mean complexity of an ensemble) would be incomputable in certain cases. By requiring the complexity of a graph to be bounded, we ensure that these quantities will be well-defined for all finite graph sets.

3 Four "Reasonable" Axioms (and Their Implications)

Having set forth four simple properties which we take to be fairly uncontroversial requirements of any graph complexity measure, we now proceed to examine four more axioms whose motivations (and implications) are less trivial. While all of these axioms are "reasonable" in the sense that one can make a fairly strong theoretical argument for each, none is unassailable; as we shall see, measures satisfying all of these axioms jointly are only of limited theoretical interest.

The first "reasonable" requirement which we will suggest for the complexity of a graph is that of a floor value: intuitively, it seems unreasonable for any graph to be unboundedly simple, just as it seems unreasonable for any graph to be unboundedly complex. Furthermore, it would seem intuitively reasonable to suppose that the graph of minimum complexity should be the single isolate $(K_1, \text{ under our assumption of looplessness}^6)$. While one could imagine a complexity measure in graphs other than K_1 would be of minimum complexity, it is difficult to see how any complexity measure could legitimately assign a *lower* complexity to any other graph than to K_1 ; hence, we let this graph define the floor complexity value:⁷

⁶Note that throughout this document the symbols K_n , N_n , P_n , and C_n will be used to refer to the complete graph, null graph, path, and cycle on n vertices, respectively.

⁷Recall that we are considering only cases in which |V(G)| > 0; hence, K_1 is truly the minimum possible graph in terms of cardinality.

Axiom 5 (Floor Value) $C(G) \ge C(K_1) \ \forall \ G$.

The second "reasonable" requirement which is suggested for a graph complexity measure is that of insensitivity to nodal labels. Intuitively, graph complexity would seem to be a property of the underlying structure of a graph: it should not be possible to change a graph's complexity simply by rearranging (or relabeling) the members of its node set⁸. Formally, then, we express this constraint with the following axiom:

Axiom 6 (Labeling Insensitivity) $C(L_1(G)) = C(L_2(G)) \forall L_1, L_2, G$.

Note that Axiom 6 is not trivial: it potentially excludes a number of measures which operate on vector representations of adjacency matrices (and which are thus subject to permutation effects). As we shall see, the simple Lempel-Ziv measure does not satisfy this axiom (though a variant suggested by Butts (1999) does).

While one could imagine some theoretical (or pragmatic) basis for their rejection, Axioms 5 and 6 are clearly fairly strongly motivated. The third of our "reasonable" axioms has a theoretical motivation as well, but it seems much more likely to generate controversy (both directly, and because of its implications). This requirement, which we shall call monotonicity, states that any graph must be at least as complex as its most complex subgraph. The intuition here follows clearly if one thinks of a social network as an aggregate made up of "elements" built from the "raw materials" of arcs and nodes. Just as we would find it odd to think of a large machine as being simpler than its most complex component, or a book as being simpler than its most complex sentence, it seems problematic to think of a graph as being simpler than its most complex subgraph. Thus, our requirement of monotonicity enforces the notion that a graph may not be simpler than its component parts⁹; more formally:

Axiom 7 (Monotonicity) $C(G) \ge C(H) \ \forall \ H \subset G$.

We shall see the problematic side of monotonicity presently. For now, however, it suffices to point out that any directed graph on n nodes is a subgraph of K_n , and hence a trivial consequence of Axiom 7 is that the cliques must be of maximal complexity. While this is certainly not out of line with a "complexity as cardinality" point of view, it seems somewhat counterintuitive to think of as uniform a structure as the complete graph as being especially complex. Clearly, we seem to have uncovered a tension within our intuitive view of complexity; as we will see, this is indeed the case.

For the fourth (and last) of our "reasonable" assumptions, we consider the complexity of graphs and their complements. In particular, we shall hold it as desirable that any complexity measure assign the same value to a digraph as to its complement. At first blush, this may seem to be an odd notion: why should it be that the complexity induced by a set of arcs (holding nodes

⁸Or, equivalently, by permuting the rows and columns of its adjacency matrix.

⁹The term "component" is here used in its nontechnical sense.

constant) is the same as that induced by the associated set of "holes"? The intuition here may be most easily appreciated by considering the adjacency matrix of a particular digraph. Traditionally, it is standard to code arcs which are present with 1s, and those which are not with 0s; what if, however, we were to code in the opposite fashion, with 0s for present arcs and 1s for those what are absent? If our complexity measure obeys complementarity, then changing the way we code the graph cannot alter its complexity — it will be the same in either case. If our measure does not obey complementarity, however, then observed graph complexity will be dependent upon the way in which we code our relationships. Like labeling dependence, this opens the door for arbitrary methodological problems, and runs counter to our intuition that complexity should be insensitive to coding issues.

Axiom 8 (Complementarity) $C(G) = C(\overline{G}) \forall G$.

As with Axiom 7, Axiom 8 is not immune to criticism. Clearly, one may object to equating present and absent arcs: in many substantive theoretical contexts, one is concerned with physical processes (e.g., information or disease transfer, monetary exchange, initiation of aggressive behavior) which operate solely on arcs which are present, and notions of complexity which are relevant to such processes may be asymmetric with respect to the presence/absence of ties¹⁰. Similarly, some of the trivial implications of complementarity are not necessarily intuitive. Under complementarity, for instance the complete and null graphs on n nodes must be of equal complexity; likewise, the path on three nodes must be of equal complexity to $K_1 \cup K_2$. While there are good reasons to expect a graph complexity measure to conform to Axiom 8, then, one could also object that this axiom forces an unwarranted symmetry between ties and holes.

3.1 Consistency of the Eight Basic Axioms

Having set out four initial (uncontroversial) axioms and four additional (perhaps more controversial) axioms, the question now arises as to whether or not there exists some complexity measure (or family thereof) which simultaneously satisfies our eight suggested axioms. This, of course, is equivalent to asking whether or not the axioms in question are consistent. As it happens, it is easy to show that if the axioms are consistent, any measure which satisfies them must satisfy some fairly stringent requirements:

Theorem 1 A complexity measure C(G) satisfies axioms 1-8 only if $C(G_1) = C(G_2) \forall G_1, G_2 : |V(G_1)| = |V(G_2)|$ and $C(G_1) \geq C(G_2) \forall G_1, G_2 : |V(G_1)| > |V(G_2)|$.

Proof: By Axiom 7,
$$C(G) \leq C(K_{|V(G)|}) \forall G$$
. By Axiom 8, however, $C(N_{|V(G)|}) = C(\overline{N_{|V(G)|}}) = C(K_{|V(G)|})$, and by Axiom 7 $C(N_{|V(G)|}) \leq C(G)$. Therefore,

¹⁰For instance, the overhead associated with keeping track of information flow within a given network may grow with the number of realized ties between actors.

 $C(K_{|V(G)|}) \leq C(G) \leq C(K_{|V(G)|})$, which implies that $C(G_1) = C(G_2) \forall G_1, G_2 : |V(G_1)| = |V(G_2)|$. Further, note that $C(G_1) = C(K_{|V(G_2)|})$ and $C(G_2) = C(K_{|V(G_2)|}) \forall G_1, G_2 : |V(G_1)| < |V(G_2)|$, however, $K_{|V(G_1)|} \subset K_{|V(G_2)|}$ which implies (by Axiom 7) that $C(K_{|V(G_2)|}) \leq C(K_{|V(G_2)|})$; therefore, $C(G_1) \leq C(G_2) \forall G_1, G_2 : |V(G_1)| < |V(G_2)|$. \square

Note that we have not, with Theorem 1, shown that any complexity measure actually satisfies our initial axioms. Clearly, however, if such a measure exists it will be to within an order-preserving transformation of |V(G)|. This is an interesting, if perhaps distressing, result: the only way we can simultaneously satisfy complementarity and monotonicity is to have a complexity measure which depends solely on the size the of graph in question! To obtain anything more subtle, we must be willing to relax one or both constraints. While we shall proceed to an examination of the status of a number of existing complexity measures with respect to this particular choice presently, it behooves us first to finish with the matter at hand. As hinted above, the necessity of Theorem 1 can also be turned into a sufficiency; in Theorem 2 below, it is shown that the measure C(G) = |V(G)| satisfies our eight proposed complexity axioms.

Theorem 2 The complexity measure C(G) = |V(G)|, together with the set of loopless digraphs, provides a model for axioms 1-8.

Proof: |V(G)| is a finite real number which is defined $\forall G$; thus, Axioms 1, 3, and 4 are clearly satisfied (along, trivially, with Axiom 2). Note that the minimum value of |V(G)| is obtained on K_1 , thereby satisfying Axiom 5. Trivially, |V(G)| is not dependent on nodal labels, hence Axiom 6 is satisfied as well.

To see that C(G) = |V(G)| satisfies monotonicity (Axiom 7), we proceed to consider two separate cases. First, we take two digraphs $G_1, G_2 : |V(G_1)| < |V(G_2)|$. Because $G_2 \not\subset G_1$ we need consider only the possibility that $G_1 \subset G_2$. If this is the case, then Axiom 7 requires that $C(G_1) \leq C(G_2)$. Since $C(G_1) = |V(G_1)|$, $C(G_2) = |V(G_2)|$, and $|V(G_2)| > |V(G_1)|$, this is obviously satisfied. In the alternative case in which $|V(G_1)| = |V(G_2)|$, it follows that $C(G_1) = C(G_2)$, and thus clearly Axiom 7 must be satisfied in this case as well (as both $C(G_1) \leq C(G_2)$ and $C(G_2) \leq C(G_1)$ are true).

Finally, we consider complementarity (Axiom 8). Note that C(G) = |N| is constant for all graphs of equal order; hence, Axiom 8 is automatically satisfied. As axioms 1-8 are true for C(G) = |V(G)| on all loopless digraphs, it follows

As axioms 1-3 are true for C(G) = |V(G)| on all loopless algraphs, it follows that C(G) = |V(G)| (together with the set of loopless digraphs) provides a model for these axioms. \square

From Theorem 2, the consistency of the proposed axioms follows trivially:

Correlary 1 Axioms 1-8 are consistent.

Proof: By Theorem 2, the complexity measure C(G) = |V(G)| provides a model for axioms 1-8. It therefore follows that axioms 1-8 are consistent. \square

At this point, then, we have shown that considering a number of fairly simple constraints on the set of acceptable graph complexity measures leads us to three obvious categories: non-monotonic measures (in which graphs may be less complex than their subgraphs), non-complementary measures (in which graphs need not have the same complexity as their complements), and measures of the |V(G)| family (which satisfy both constraints, but whose practical value is very limited). Is there a simple relaxation of one or more of these conditions which will allow us to preserve at least some of the properties which motivated them without limiting us to such a narrow family of measures? As it happens, such relaxations exist; we shall consider one such modified axiom in the next subsection of this essay.

3.2 A Relaxation of Monotonicity

As we have seen, the simultaneous assumption of monotonicity and complementarity implies a family of complexity measures which are structurally degenerate (and hence uninteresting for most applications). Given, however, that both of these properties have reasonable motivations, is there some way in which one could partially retain them without accepting degeneracy? Here, we demonstrate one such alternative: by partially relaxing the monotonicity condition, we can obtain interesting complexity measures which nevertheless conform to a number of our intuitive expectations.

To consider which alternatives to Axiom 7 might be appropriate, it is instructive to examine Theorem 1. Note that the structural degeneracy of the resulting family of complexity measures is a consequence of the interaction of the requirement that K_n be of maximal complexity (a simple implication of Axiom 7) with the requirement that K_n have the same complexity as N_n (which follows from Axiom 8). To avoid this result, any change to monotonicity must be such that K_n 's complexity is no longer required to be maximal¹¹ – since this was a natural consequence of our "building block" intuition, however, we shall have to found our new axiom on a somewhat different basis. One possibility for such a modification lies in thinking of induced subgraphs, rather than subgraphs per se, as being the subcomponents¹² of larger social structures. Such an intuition is sensible if one takes the connections between nodes as being "given" due to exogenous factors, but regards the selection of an observed node set as being subject to variation¹³; as this situation is not uncommon in network analysis, the associated intuition would seem fairly reasonable. Formally, then, we state our restricted monotonicity as follows:

Axiom 9 (Sample Monotonicity) $C(G) \ge C(G[S]) \ \forall \ S \subset V(G)$.

The motivation for calling this restricted version of monotonicity "sample" monotonicity lies in its practical implication; in particular, the property implies

¹³Although this is only one interpretation.

¹¹Without introducing contradictions. This will, however, not be a problem in our case.

¹²As before, we use this term in its nontechnical sense.

that the complexity of any network sample from a larger graph will provide a lower bound on the complexity of the entire (unobserved) structure. The practical benefit of this is obvious: as the networks in which we are interested are often supersets of those we actually observe, being able to draw a guaranteed inference regarding the complexity of the larger structure is clearly useful¹⁴. Note, too, that sample monotonicity is clearly a weaker condition than the form of monotonicity described in Axiom 7. Trivially, any complexity measure which satisfies the latter must satisfy the former, as an induced subgraph of a digraph is (by definition) also a subgraph per se. Similarly, the converse of this statement is false: a complexity measure could satisfy Axiom 9 while giving C_4 a higher complexity than K_4 for (since all induced subgraphs of K_4 are cliques), which would violate Axiom 7. Monotonicity thus implies, but is not implied by, sample monotonicity.

Given that we can define an apparently intuitive relaxation of monotonicity, what sort of measures does it imply? In particular, how does it interact with complementarity? Is it consistent with the other axioms? To answer these questions, we demonstrate a new graph complexity measure which satisfies both conditions simultaneously.

One way in which one might think of the complexity of a graph is with respect to the number of "types" of structures found within it. A clique, for instance, is relatively simple in this sense, as it contains only additional cliques; by contrast, a random graph of equivalent size will often contain a wide range of varying structures, and hence will be more complex (given this notion of complexity). Of course, one might ask what is meant here by "containment". Obviously, the notion of subgraph is not consistent with the above – as has been noted, a clique contains all graphs of equal or lower order as subgraphs – but that of containment vis a vis *induced* subgraphs is indeed permissible. To construct our new complexity measure, then, we shall take the complexity of a given directed graph to be the size of the set of all distinct (i.e., non-isomorphic) induced subgraphs contained within it; formally:

Definition 1 (Induced Subgraph Complexity) Let S_G be the set of all distinct induced subdigraphs of directed graph G. The induced subgraph complexity of G, $C_{IS}(G)$, is given by $C_{IS}(G) = |S_G|$.

Is this notion of complexity consistent with our previous axioms (absent monotonicity)? As it happens, the answer is yes, as is shown in Theorem 3 below.

Theorem 3 The induced subgraph complexity measure, C_{IS} , together with the set of loopless digraphs, provides a model for axioms 1-6, 8, and 9.

Proof: By Definition 1, $C_{IS}(G) = |S_G|$, where S_G is the set of distinct induced subgraphs of G. $|S_G|$ is a finite real number which is defined $\forall G$; thus,

¹⁴Of course, a lower bound is just that – we cannot know whether the larger structure is substantially *more* complex without observing it. Nevertheless, a lower bound is better than no bound at all.

Axioms 1-4 are clearly satisfied. Because $|S_G|$ takes its minimum value (0) on K_1 , Axiom 5 is also satisfied, as is Axiom 6 (due to the fact that the number of distinct subgraphs of G is a labeling independent property).

To verify that C_{IS} satisfies complementarity (Axiom 8), first assume the contrary. Then, by definition, $\exists G: C_{IS}(G) \neq C_{IS}(\overline{G})$. Without loss of generality, we take $C_{IS}(G) > C_{IS}(\overline{G})$. By Definition 1, this implies that \exists induced subgraphs $s_1, s_2 \subset G: s_1 \not\cong s_2$ and $\overline{s_1} \cong \overline{s_2}$ (where \cong signifies the relation "is isomorphic to"). However, $\overline{H_1} \cong \overline{H_2}$ implies $H_1 \cong H_2 \vee$ digraphs H_1, H_2 ; therefore our assumption implies a contradiction, and hence it must be the case that C_{IS} satisfies complementarity.

Finally, we show that sample monotonicity (Axiom 9) is satisfied. Assuming that C_{IS} fails to satisfy Axiom 9 implies that $\exists S \subset N(G) : C_{IS}(G[S]) > C_{IS}(G)$. Let H_G be the set of all distinct induced subgraphs of G, and let $H_{G[S]}$ be the set of all distinct induced subgraphs of G[S]. By the definition of subgraph, $H_{G[S]} \cup H_{G[N(G)-S]} \subseteq H_G$; therefore, $|H_G| \ge |H_{G[S]}| + |H_{G[N(G)-S]}| - |H_{G[S]} \cap H_{G[N(G)-S]}| \ge |H_{G[S]}|$. This contradicts our initial assumption, and thus C_{IS} must satisfy sample monotonicity.

As axioms 1-6, 8, and 9 are true for C_{IS} on all loopless digraphs, it follows that C_{IS} (together with the set of loopless digraphs) provides a model for these axioms. \square

Thus, induced subgraph complexity provides a working example of a non-trivial measure which satisfies the majority of our intuitive requirements for a measure of complexity. In demonstrating this, the above theorem clearly implies the consistency of these axioms as well:

Correlary 2 Axioms 1-6, 8, and 9 are consistent.

Proof: By Theorem 3, the induced subgraph complexity measure provides a model for axioms 1-6, 8, and 9. It therefore follows that axioms 1-6, 8, and 9 are consistent. \square

What, now, of the question of behavior? While a full examination of the behavior of the induced subgraph complexity measure is beyond the scope of this paper, it is perhaps useful to compare this measure with some number of other measures found in the literature. In this case, we illustrate the behavior of the induced subgraph complexity measure by comparison with the orbit information measure of Mowshowitz (1968a, 1968b, 1968c) and the role complexity of Everett (1985) on all distinct graphs of order four¹⁵. The presentation here (see Table 1 below) follows that of Everett (1985), and the data for I(G) (orbit information) and $R_c(G)$ (role complexity) is taken from the same source.

¹⁵Since all three measures satisfy complementarity, only half of the distinct graphs are shown; complements of those listed will be of identical complexity. Note that only simple graphs are considered here, in order to maintain comparability with Mowshowitz (1968a) and Everett (1985).

Table 1: C	omparison of	Three Cor	mplexity Measures
G	I(G)	$R_c(G)$	$C_{IS}(G)$
$4K_1$	0	0	3
$2P_2$	0	$\frac{2}{3}$	4
$C_3 \cup K_1$	$2 - \frac{3}{4} \log_2 3$	$\frac{3}{4}$	5
$P_2 \cup 2K_1$	1	<u>5</u>	5
$P_3 \cup K_1$	$1\frac{1}{2}$	$\frac{11}{12}$	6
P_4	1	$\frac{11}{12}$	5

It is interesting to note that, as Table 1 indicates, the behavior of C_{IS} is distinct from that of the other two measures even when considering very small graphs. While both the null graph on four vertices and union of two disconnected dyads are considered to be of minimal complexity under the orbit information measure, the latter is clearly more complex with respect to the induced subgraph measure. Likewise, both the path on four vertices and the union of the three vertex path with a single isolate are considered to be equally complex under the role complexity measure, despite the fact that the former exhibits less induced subgraph complexity. While all three measures clearly show a general trend towards identifying certain kinds of structures (e.g., long paths) as being more complex than others (e.g., cliques), each corresponds to a distinct notion of complexity, and as such there is not total agreement between them. Identification of the conceptual divisions between these measures is pursued in the next section; systematic illustration of the behavioral implications of each is a topic for future research.

4 Existing Measures of Structural Complexity, by the Axioms

While we could continue to add and/or modify axioms in an a priori fashion, creating an increasing number of complexity measure classes (and measures), we shall now follow a somewhat different pattern: having laid out these initial axioms, we shall proceed to examine how a number of existing measures of graph complexity may be classified, adding axioms as needed to discriminate between measures. By so doing, we hope to illustrate the conceptual significance associated with choosing one measure over another for a given application.

4.1 Some Sample Complexity Measures

As has been noted, the literature on complexity utilizes a wide variety of distinct notions of what it means for a system, structure, or algorithm to be "complex"; even in the more restricted case of graph complexity, a number of possible measures are available. Even excluding the notion of graph dimensionality (Roberts, 1969; Guttman, 1977; Freeman, 1980; Everett, 1984) as a suitable complexity

construct¹⁶, the set of potential measures is quite large. Here, we have chosen to focus on six existing complexity measures (in addition to induced subgraph complexity, which was introduced earlier). These measures are as follows:

Cardinality of the vertex set: As we have already seen, the cardinality of the vertex set supplies a model for our eight basic axioms; as such, it is a natural candidate for inclusion. Substantively, graph size is also a widely used intuitive notion of complexity in organizational and mental model contexts. In the text which follows, vertex set cardinality will be represented in the usual fashion (|V(G)|).

Cardinality of the arc set: Along with graph size, it is intuitive in many contexts to presume that a structure is more "complex" insofar as there are more connections between its component parts. As with vertex set cardinality, arc set cardinality is represented via the typical notation of |E(G)|.

Source entropy of the arc set: A natural frame for considering the information content (and arguably the complexity) of a graph is that of a context in which we imagine the graph to be transmitted on an arc-by-arc basis to some receiver. In such a situation, the expected information content of each arc (in bits) corresponds to the related first order Shannon entropy, or $-(d\log_2 d + (1-d)\log_2 (1-d))$, where d is the density of the digraph in question. The source entropy of the arc set scales the asymptotic maximum Lempel-Ziv complexity (Kaspar and Schuster, 1987), and is intuitively related to the expected degree of graph compressibility¹⁷. Here, we represent this measure by h(G).

Orbit information: Mowshowitz (1968a, 1968b, 1968c) defines the complexity of a graph in terms of the expected information content of an optimally encoded signal regarding the orbit membership of a randomly chosen vertex. Formally, for a digraph G with orbits $A_1, A_2, ...A_{|Orb(G)|}$, the orbit information (I(G)) is given by $I(G) = -\sum_{A \in Orb(G)} \frac{|A|}{|V(G)|} \log_2 \frac{|A|}{|V(G)|}$. The use of orbit

information as a measure of structural complexity developed originally out of early work by Morowitz (1955), Rashevsky (1955), Trucco (1956), and others in topological biology, where its intuitive motivation involved quantifying the difficulty of reconstructing an organism from (chemically) structurally defined constituents; the parallels between this notion and group or organizational structurally

¹⁶This approach was introduced to the social network literature by Freeman (1980), but has been criticized by Everett (1985) on the grounds that dimensionality is conceptually distinct from complexity and that the particular measures of dimension offered in previous work yield intuitively unappealing results. Here, we have chosen to focus on more traditional graph and information theoretic approaches, and do not treat dimensional measures of complexity; it should be noted, however, that we do not dismiss the notion that dimensional measures may be of conceptual or practical value.

¹⁷More formally, an upper bound on the length of the optimal encoding of a graph is given by $h(G)|V(G)|^2$.

ture are obvious.

Role complexity: Everett (1985) defines a notion of graph complexity based on the size of a graph's automorphism group. Though related to the measure of Mowshowitz (1968a, 1968b, 1968c), Everett's role complexity $(R_c(G))$ more directly addresses the degree of exchangability between positions¹⁸. (Alternatively, it can also be understood in terms of the chance of drawing an automorphism from the set of all node labelings under a uniform sampling condition.) Role complexity is given by $R_c(G) = 1 - \frac{|AutG|}{|V(G)|!}$, where AutG is G's automorphism group.

Induced subgraph complexity: Induced subgraph complexity (the number of distinct induced subgraphs of a given graph) was introduced in Definition 1 above. Represented by $C_{IS}(G)$, this measure can be interpreted as the number of different substructures composing a given graph (where "substructure" is given to mean an arbitrary set of vertices and all associated arcs); it is hence a measure of what might be called "structural diversity".

Lempel-Ziv complexity: Lempel and Ziv (1976) present a measure of algorithmic complexity for finite sequences which can be shown to behave like the Kolmogorov-Chaitin complexity (Kolmogorov, 1965) in certain respects (Lempel and Ziv, 1976). Subsequent work by Kaspar and Schuster (1985) demonstrated the applicability of the measure to dynamic systems, and estimated its convergence properties on simulated data. Butts (1999) indicated a means of calculating the Lempel-Ziv complexity on suitably encoded digraphs, and showed that the measure was strongly related to the presence of nontrivial structural equivalence classes; an empirical examination of a range of network data sets suggested that the Lempel-Ziv complexity of actual relational networks is generally quite high. Due to the length of the definition (and the fact that it is presented in Butts (1999)), it will not be given in full here. In brief, however, $C_{L-Z}(G)$ is the number of insertion operations required by a copy/insertion machine to produce a given n-ary sequence, and provides an index of the algorithmic complexity of the sequence in question.

Having described each of the measures which will be used in the remainder of the paper, we may now proceed to the problem of axiomatic discrimination between measures of graph complexity.

4.2 Axiomatic Discrimination Between Measures

One use of a formal logical framework is to facilitate the development of constructs which can be shown to obey certain desirable properties. Another is to identify conceptual distinctions between constructs, and to create a system for

¹⁸It is also related to algorithmic complexity (see Butts (1999) for a discussion of algorithmic complexity on social networks), due to the fact that the presence of nontrivial automorphism classes implies compressibility.

their classification. Earlier in this paper, the former usage of complexity axioms was deployed; now, we shall consider the latter. Given the set of measures at our disposal, will the axioms so far introduced allow us to differentiate between them? To find out, we shall now proceed to show which axioms are (and are not) satisfied by each of the measures in question.

First, we shall consider the axioms 1-5. As we have seen, the measures already examined have been found to satisfy these requirements; in fact, it happens that *all* of the measures under consideration obey these axioms, and hence they will be ignored for the remainder of the analysis. (Detailed proofs are not presented, due to the fact that they follow the form of the proofs for Theorems 2 and 3, but are available upon request.)

Next, we note that two of our measures -|V(G)| and $C_{IS}(G)$ - have already been dealt with in Theorem 2 and Theorem 3, respectively, with the exception of the the proof of sample monotonicity for |V(G)|. For this last, recall from the proof of Theorem 2 that |V(G)| satisfies monotonicity (Axiom 7). As was observed earlier, any measure which satisfies monotonicity must also satisfy sample monotonicity; therefore, it obviously follows that |V(G)| satisfies sample monotonicity as well. Thus, |V(G)| satisfies all nine of our current axioms, while $C_{IS}(G)$ satisfies Axioms 6, 8, and 9 in addition to the first five (which are satisfied by all measures under consideration).

For |E(G)|, we state the following theorem:

Theorem 4 |E(G)| satisfies Axioms 6, 7, and 9; |E(G)| does not satisfy Axiom 8.

Proof: Trivially, the cardinality of the arc set of a loopless digraph G is labeling invariant; thus, Axiom 6 is obviously satisfied.

To verify that |E(G)| obeys monotonicity (Axiom 7), recall that the arc set of a subdigraph is a subset of the arc set of the original digraph. It therefore follows that $|E(H)| \leq |E(G)| \ \forall \ H \subset G$; thus, Axiom 7 is satisfied. (Since monotonicity implies sample monotonicity, Axiom 9 is satisfied as well.)

Finally, we note that $|E(K_n)| > |E(\overline{K_n})| \ \forall \ n > 1$. Thus, |E(G)| fails to satisfy Axiom 8. \square

As shown, arc set cardinality satisfies all axioms but complementarity; this result is quite intuitive when one considers the original rationale behind the measure. If a structure is more complex to the extent to which it has more internal connections, then it obviously follows that arcs and "holes" are not synonymous, and the measure in question would not be expected to obey complementarity. E(G), then is an example of a "compromise measure" in the sense discussed earlier: it avoids degeneracy while maintaining monotonicity by forgoing complementarity.

The other side of this compromise is illustrated below by source entropy, for which we state:

Theorem 5 h(G) satisfies Axioms 6 and 8; h(G) does not satisfy Axiom 7 or Axiom 9.

Proof: Recall that h(G) can be expressed as $-(d \log_2 d + (1-d) \log_2 (1-d))$, where d is the density of G (more precisely, $d = \frac{|E(G)|}{|V(G)|^2 - |V(G)|}$). Thus, h(G) depends only on the cardinalities of the vertex and arc sets of G; these are labeling independent properties, and thus h(G) must satisfy Axiom 6. Also, observe that since $d(\overline{G}) = 1 - d(G)$, substitution gives us $h(\overline{G}) = -((1-d)\log_2(1-d) + d\log_2 d) =$ h(G). Therefore, it follows that h(G) obeys Axiom 8 as well.

That h(G) cannot satisfy Axiom 7 is immediate from the observation that h(G) takes its maximum (1) when d = 0.5, and is at its minimum (0) for d = 1. Thus $h(G) > h(K_{|V(G)|})$ for G : d = 0.5, but $G \subset K_{|V(G)|}$. This violates Axiom 719.

Lastly, a counterexample to the assertion that h(G) satisfies sample monotonicity (Axiom 9) is provided by $G: V(G) = \{a, b, c, d\}, E(G) = \{(a, b)\}$ and vertex subset $S = \{a, b\}$. In this case, h(G) = 0.414 which is less than h(G[S]) = 1, which violates the conditions of Axiom 9. \square

In the case of h(G), then, it is monotonicity which is sacrificed to preserve complementarity. Here again, this is sensible when one considers the basis of the measure itself: the source entropy of the arc set is a measure of the expected uncertainty regarding the existence of any given arc without regard to its position, and hence treats arcs and holes symmetrically. Uncertainty per arc is not related in any straightforward way to the particular network sample in question²⁰, and thus the monotonicity axioms are violated.

These observations, interestingly, seem to generalize (to some extent) to other information-based measures of network complexity. Consider the case of orbit information:

Theorem 6 I(G) satisfies Axioms 6 and 8; I(G) does not satisfy Axiom 7 or Axiom 9.

Proof: By definition, two vertices $i, j \in V(G)$ are members of the same orbit iff \exists a relabeling l of the vertices of G: l(i) = j and $l(G) \cong G$; relabeling, hence, cannot affect orbit structure, and I(G) must satisfy Axiom 6.

To verify that I(G) satisfies Axiom 8, we first assume the contrary. If I(G)does not obey Axiom 8, it follows that $\exists i, j \in V(G)$ and relabeling l of V(G): $l(i) = j, l(G) \cong G, \text{ and } l(\overline{G}) \not\cong \overline{G}.$ However, it is also the case that $G \cong H$ implies $\overline{G} \cong \overline{H} \ \forall$ digraphs G, H; hence, $l(G) \cong G$ implies $l(\overline{G}) \cong \overline{G}$, and (by contradiction) I(G) must satisfy Axiom 8.

For brevity, we shall now demonstrate that I(G) fails both monotonicity (Axiom 7) and sample monotonicity (Axiom 9) by showing that I(G) does not satisfy

¹⁹This could also be proven simply by noting that the weaker condition of sample monotonicity is violated (see below). The separate proof of this proposition is given to aid intuition regarding the behavior of h(G).

20 Ceterem paribus, and providing that the sample is not degenerate.

the latter (and therefore cannot obey the more stringent condition). Consider the graphs $P_2 \cup 3K_1$ and $P_2 \cup 2K_1$ (taking all arcs as reciprocal). The orbit information values of these graphs are $I(P_2 \cup 3K_1) = 0.971$ and $I(P_2 \cup 2K_1) = 1$, respectively; however, $P_2 \cup 2K_1$ is an induced subgraph of $P_2 \cup 3K_1$, and Axiom 9 (along with Axiom 7) is therefore violated. \square

The pattern of I(G) is identical to that of h(G): only the monotonicity axioms are violated. This similarity, as it happens, is not entirely accidental. In general, any information measure defined on a partition p_1, p_2, \dots of graph features (e.g., orbits, dyadic states, etc.) will violate Axioms 7 and 9 if for any graph G and partition i it is possible to create a new graph H such that $\exists S \subset V(H) : G = H[S] \subset H, |p_i(H)| > |p_i(G)|, \text{ and } |p_i(H)| \leq |p_i(G)| \forall j \neq i.$ The precise nature of the graph feature per se is irrelevant; if it is possible to add members to exactly one partition simply by adding vertices and associated edges to a graph, then one may ultimately obtain a violation of sample monotonicity (and therefore monotonicity as well) simply by repeating this process a sufficient number of times²¹. This follows directly from the fact that the information measure is 0 for the uniform distribution and approaches unity as an increasing proportion of probability mass is placed on a small number of partitions²². While it is conceivable that one could choose a set of features which would not allow such concentration of cases within a particular partition, one suspects that most interesting features will have this property; thus, a wide range of graph information measures, at the very least, will not be suitable for applications requiring the monotonicity axioms.

As we have seen, then, it is possible to satisfy the tension between monotonicity and complementarity by completely discarding one or the other. Similarly, some measures compromise by obeying only the more relaxed form of monotonicity. $C_{IS}(G)$, discussed, above, is one such measure; another, as it happens, is role complexity, as is shown by the theorem below.

Theorem 7 $R_c(G)$ satisfies Axioms 6, 8 and Axiom 9; $R_c(G)$ does not satisfy Axiom 7.

Proof: Because (within graphs of a given order) role complexity depends only on the automorphism group of G, the arguments used in the proof of Theorem 6 regarding Axioms 6 and 8 also apply in the case of $R_c(G)$; thus, by the earlier proof, Axioms 6 and 8 are satisfied.

To show that $R_c(G)$ satisfies Axiom 9 (sample monotonicity), it is first useful to observe that this assertion is equivalent to the statement that $\exists G, v \in V(G)$: $\frac{|AutG|}{|AutG-v|} > |V(G)|$. This follows from the fact that $R_c(G) = 1 - \frac{|AutG|}{|V(G)|!}$, and that a violation of sample monotonicity would require there to exist some graph whose role complexity could be decreased by adding a new vertex²³.

²¹For instance, we obtained the violations of Theorem 6 by adding isolates.

²²Note that, as this suggests, the stated condition is sufficient but not necessary to guarantee a violation of the monotonicity axioms.

²³Note that this last is a necessary and sufficient condition for violation of Axioms 9.

Having reframed the problem in this fashion, we now demonstrate that, in fact $|\exists G, v \in V(G): \frac{|AutG|}{|AutG-v|} > |V(G)|^{24}$. Let $Fix(v,G) = \{g: g \in Aut(G), g(v) = v\}$, or the set of automorphisms of G fixing v, and let $Orb(v,G) = \{u: \exists g: g(v) = u\}$ (or the orbit of v in g). It is a basic result of the theory of permutation groups²⁵ that:

$$\frac{|AutG|}{|Fix(v,G)|} = |Orb(v,G)| \forall v \in V(G)$$

Some simple algebra, then, gives us

$$\frac{|AutG|}{|AutG-v|} = \frac{|Fix(v,G)||Orb(v,G)|}{|AutG-v|}$$

We now observe that Fix(v,G) is a subgroup of AutG-v; therefore, $\frac{|Fix(v,G)|}{|AutG-v|} \le 1$, and hence

$$\frac{|AutG|}{|AutG-v|} \leq |Orb(v,G)|$$

Orb(v,G), however, is obviously a subset of V(G), which implies that

$$\frac{|AutG|}{|AutG-v|} \le |V(G)| \forall v \in V(G)$$

Thus, $\exists G, v \in V(G) : \frac{|AutG|}{|AutG-v|} > |V(G)|$, and by our initial argument, $R_c(G)$ must satisfy Axiom 9.

Finally, we note that $|AutK_n| = n!$, and hence $R_c(K_n) = 0$; there are, however, subgraphs of K_n (such as P_4) with higher complexity (in this case, $\frac{11}{12}$), which violates Axioms 7. \square

Although it is quite non-obvious that $R_c(G)$ should satisfy sample monotonicity, it nevertheless does; in fact, it obeys all of our proposed axioms, with the exception of full monotonicity. That role complexity has this property is shown in the proof of Theorem 7 to follow from the fact that the normalizing term of $R_c(G)$ must grow at least as fast as the size of AutG for all G. Thus, once a particular graph "falls behind" in the sense of having a small automorphism group, there is no way for it to "catch up" by adding nodes and associated arcs. This fact also suggests that the population of large structures (even those with

²⁴I am indebted to Martin Everett and John Boyd for this result.

²⁵This is a version of a theorem usually credited to Burnside.

nontrivial automorphism classes) will tend to be concentrated towards complexity 1²⁶, and that finite random graphs should approach maximum complexity quite rapidly under a standard process of evolution by addition of arcs.

The last of the measures which we shall here consider differs rather substantially both in the manner of its definition and in the particular axioms which it does (and does not) satisfy:

Theorem 8 $C_{L-Z}(G)$ satisfies Axioms 8 and 9; $C_{L-Z}(G)$ does not satisfy Axioms 6 or 7.

Proof: As noted, the Lempel-Ziv is given by the number of insertion operations required by a (copy/insertion) production process which regenerates the sequence whose complexity is under evaluation. Following Butts (1999), we here assume that the complexity of a digraph G is taken to be the L-Z complexity of the vector formed by a concatenation of the rows of the G's adjacency matrix $(A)^{27}$.

Given this, the L-Z complexity is defined on an arbitrary alphabet of symbols; as the adjacency matrix of \overline{G} is merely the adjacency matrix of G with 0s and 1s exchanged, it follows that $C_{L-Z}(G)$ must obey complementarity (Axiom 8).

In order to demonstrate that $C_{L-Z}(G)$ satisfies Axiom 9, first note that $C_{L-Z}(SQ) \geq C_{L-Z}(S)$ for all vectors S,Q with concatenation SQ^{28} . Now consider a digraph H with adjacency matrix A_H . If we add some set of vertices V_{G-H} to H (forming a digraph G of which H is an induced subgraph), then the adjacency matrix A_G will consist of four submatrices: $A_{V(H)xV(H)}$, $A_{V(H)xV(G-H)}$, $A_{V(G-H)xV(H)}$, and $A_{V(G-H)xV(G-H)}$ (where A_{axb} represents the adjacency matrix associated with the set of arcs from vertex set a to vertex set b). Note that, by our earlier observation, $C_{L-Z}(A) \geq \max_{i \in Rows(A)} C_{L-Z}(A_i)$.

Thus, $C_{L-Z}(A_G) \geq C_{L-Z}(A_{V(H)xV(G-H)}A_{V(H)xV(H)}) \geq C_{L-Z}(A_H)$, and Axiom 9 is satisfied.

An example of a violation of Axiom 6 has already been given by Butts (1999), in showing that the Lempel-Ziv complexity of the isomorphic structures

 $^{^{26}}$ Indeed, since |AutG|=1 for almost all graphs, it follows that almost all graphs are of maximum complexity under $R_c(G)$; that this is also true (for similar reasons) of algorithmic complexity underscores the connection between the two.

²⁷Butts (1999) also discusses alternative encoding schemes – such as dyadic encoding – as well as a means of estimating an L-Z complexity measure for unlabeled graphs. Here, we use only the initial (labeled, arc encoded) formulation, and our conclusions apply only to this version of the complexity measure.

²⁸We shall here take C_{L-Z} to be defined on graphs, matrices, and sequences, using the encoding assumption mentioned above.

0	1	1	1	0	0	0	0]	0	0	1	0	1	0	1	0
1	0	1	1	0	0	0	0		0	0	0	1	0	1	0	1
1	1	0	1	0	0	0	0		1	0	0	0	1	0	1	0
1	1	1	0	0	0	0	0	and	0	1	0	0	0	1	0	1
0	0	0	0	0	1	1	1	unu	1	0	1	0	0	0	1	0
0	0	0	0	1	0	1	1		0	1	0	1	0	0	0	1
0	0	0	0	1	1	0	1		1	0	1	0	1	0	0	0
0	0	0	0	1	1	1	0		0	1	0	1	0	1	0	0

are not identical (the exact values are 10 and 11, respectively). To show violation of monotonicity, it is sufficient to note that the complete graph produces a 1-vector, which has complexity 2 (Lempel and Ziv, 1976), and that the structure

0	0	0	1
1	0	1	0
0	1	0	0
0	1	0	0

has a complexity of 5 (exhaustive history given by $0 \cdot 001 \cdot 10 \cdot 1000 \cdot 1000 \cdot 100$ – note the lack of a closing dot, indicating that no insertion was required for the final word). Since this structure is a subgraph of K_4 , it follows that $C_{L-Z}(G)$ does not satisfy Axiom 7. \square

Unlike every other measure considered here, then, the Lempel-Ziv complexity (given the assumed encoding scheme) fails to satisfy labeling independence. This is a consequence of the fact that the vector encoding is not itself labeling independent; while not every measure encoded in this fashion will necessarily be labeling dependent, it is worth bearing in mind that many will (and hence one should bear this in mind when examining measures based on such a scheme). $C_{L-Z}(G)$, on the other hand, does obey both complementarity and sample monotonicity. This is to be expected from a measure which is based on the algorithmic complexity, as it should be A) symmetric in its treatment of elements in the coding alphabet, and B) nondecreasing as the signal to be compressed increases in length. (Although a version of $C_{L-Z}(G)$ presented by Butts (1999) does satisfy labeling invariance²⁹, the original version is given here for simplicity.)

Given the above theorems, then we can clearly differentiate between a number of complexity measures. $R_c(G)$ and C_{IS} , however (along with h(G) and C_{L-Z}), satisfy the axioms, and cannot be uniquely classified on the basis of Axioms 1-9. To separate these pairs of measures, then, we introduce one final axiom:

Axiom 10 (Normalization) $\exists a, b \in \Re : a \leq C(G) \leq b \forall G$.

Normalization, in the limited sense used here, indicates the presence of a

²⁹Though this is at considerable computational cost; in general, the labeling invariant version of $C_{L-Z}(G)$ can only be estimated, due to the fact that exact computation is O(|V(G)|!).

ceiling on complexity values³⁰ in addition to the floor implied by Axiom 5. Intuitively, the difference between the notion of complexity as satisfying normalization and the notion of complexity as being unnormalized is that of whether complexity should be seen as an inherently bounded property, or whether it should be unbounded. As a simple examination of the definitions considered here indicates, this bounded notion of complexity is consistent with source entropy of the arc set (which is naturally constrained by the maximum uncertainty of a Bernoulli event) and role complexity (which is explicitly framed by Everett (1985) in terms of the (relative) degree of interchangeability among positions). None of the other complexity measures presented satisfy Axiom 10.

Having now presented a set of axioms such that each complexity measure satisfies a unique combination of postulates, we are now in a position to examine the total classification. For the five "critical" axioms (Axioms 5-10) on which the measures presented differ, Figure 1 below illustrates the area of overlap of each using a Venn diagram. Note that, as this depiction renders obvious, there are a number of cells in our classificatory system which are currently unoccupied. Insofar as these cells correspond to assumptions which are of theoretical interest, then, it may be of value to use such "empty" regions as a guide when creating new measures³¹.

³⁰Recall that the finiteness axiom only states that for each graph there be some real number greater than that graph's complexity. Normalization states that there must be a real number greater than the complexity of any graph.

³¹Of course, it is undoubtedly also the case that many cells are empty because they correspond to hypothetical measures which are inconsistent with nearly all of our notions of how a complexity measure should behave. It should not be assumed that the mere presence of a cell motivates serious consideration of the measures within.

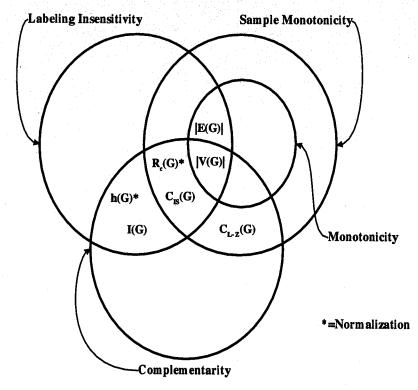


Figure 1: Discrimination by Five Critical Axioms

Another presentation of this same information is in Table 2 below; here, the proposition that each axiom is satisfied is evaluated for each measure (yielding a result of true (T) or false (F) for each measure/axiom combination). My examining the columns of this table, it is easy to get a general sense of the degree to which various axioms do or do not tend to be satisfied by the measures of complexity here examined. Labeling independence, for instance, is satisfied in nearly all cases (as is complementarity), but sample monotonicity is somewhat more divided. As can be seen, the basic combination of labeling independence, complementarity, and nonmonotonicity describes four out of the seven measures examined; labeling independence, complementarity, and sample monotonicity were satisfied by three of the seven. Clearly, then, there is some degree of consensus in terms of the assumptions on which graph complexity should be based, though there is also fair amount of disagreement even on reasonably basic matters (such as whether sample monotonicity should be obeyed). While our choice of axioms obviously affects the observed grouping, it is worth emphasizing that only Axiom 10 was selected for the purposes of differentiating among similar measures. The others here employed were derived from a priori conceptions of the notion of graph complexity or, in the case of sample monotonicity, from a relaxation of an a priori concept motivated by a tension between conceptual elements.

Table 2: Axiomatic Comparison of Graph Complexity Measures								
Measure	Lab. In. (A6)	Mono. (A7)	Comp. (A8)	Samp. Mon. (A9)	Norm. (A10)			
E(G)	${f T}$	\mathbf{T}	F	${f T}$	F			
V(G)	${f T}$	${f T}$	T	${f T}$	F			
h(G)	Τ	F	\mathbf{T}	F	${f T}$			
I(G)	T	F	${f T}$	F	F			
$R_c(G)$	T	F	${f T}$	${f T}$	T			
$C_{IS}(G)$	${f T}$	F	${f T}$	T	F			
$C_{L-Z}(G)$	F	F	T	T	F			

5 Conclusion

As we have seen, one can set out a number of "reasonable" axioms regarding the complexity of graphs, and then use these for both constructive and classificatory purposes. While complexity measures do exist which satisfy all of the suggested axioms, such measures are not interesting for most purposes; they are structurally degenerate, depending only on graph size. By relaxing certain of our requirements, it is possible to admit measures which exhibit more variability within size classes, with the particular relaxation in question determining the properties of the measures permitted. Examining one such relaxation (that of full monotonicity) allows for the definition of a new and distinct complexity measure, induced subgraph complexity, which increases with the diversity of substructures present within a graph. Considering the relaxations required to admit (and fully classify) a number of extant complexity measures, it can be seen that existing notions of graph complexity span a wide range of assumptions; although certain family resemblances between measures exist (e.g., within information-based measures), the number of widely shared properties is limited (absent certain very basic properties such as determinism and existence). Among those definitions of complexity considered here, almost all do obey the properties of labeling insensitivity and complementarity. Most do not obey monotonicity or normalization, although most do obey the less restrictive sample monotonicity property.

This study has limited itself to postulating a small number of fairly general axioms, and to examining only a handful of complexity measures, with the intent of establishing a set of basic results on which a more complete axiomatic treatment could expand. At least two directions, then, suggest themselves: first, the possibility of elaborating and refining the axiom set to allow a larger set of deductions; and, second, the classification of measures beyond those considered here. Given the current fragmentation of the subject, it would seem highly advisable for those proposing new measures of graph complexity to (at the very least) demonstrate those of the basic axioms presented here which are or are not satisfied by the measures in question, and (ideally) to show how the new measures may be derived axiomatized in their entirety. While admittedly somewhat time consuming, such a procedure would ensure that the assumptions

underlying novel measures may be clearly understood and examined, and would facilitate the comparison (and possible unification) of disparate notions of graph complexity. Though complexity appears to be a powerful and compelling concept, its study is unduly hampered by a lack of attention to foundational issues; it is hoped that the axiomatic approach presented here will aid in rectifying this situation.

6 References

Bennett, C.H. (1985). "Dissipation, Information, Computational Complexity and the Definition of Organization." In D. Pines (Ed), *Emerging Syntheses in Science*, 215-233. Redwood City, CA: Addison-Wesley.

Butts, C. (1999). "The Complexity of Social Networks: Theoretical and Empirical Findings." Center for the Computational Analysis of Social and Organizational Systems Working Paper, Carnegie Mellon University.

Cover, T. M. and Thomas, J. A. (1991) Elements of Information Theory. New York: John Wiley and Sons.

Everett, M. C. (1984). "A New Definition for the Dimension of a Graph." Unpublished conference presentation, Sunbelt Network Conference, 1984.

Everett, M. C. (1985). "Role Similarity and Complexity in Social Networks." Social Networks, 7, 353-359.

Feldman, D. P. and Crutchfield, J. P. (1998). "Discovering Noncritical Organization." Santa Fe Institute Working Paper, 98-04-026.

Freeman, L. C. (1980). "Q-Analysis and the Structure of Friendship Networks." International Journal of Man-Machine Studies, 12, 367-378.

Freeman, L. C. (1983). "Spheres, Cubes, and Boxes: Graph Dimensionality and Network Structure." Social Networks, 5, 139-156.

Guttman, L. (1977). "A Definition of Dimensionality and Distance for Graphs." In J.C. Lingoes (Ed), Geometric Representations of Relational Data. Ann Arbor: M.I. Mathesis.

Kaspar, F., and Schuster, H. G. (1987). "Easily Calculable Measure for the Complexity of Spatiotemporal Patterns." *Physical Review A*, 36(2), 842-848.

Kolmogorov, A. N. (1965). "Three Approaches to the Definition of the Concept 'Quantity of Information'." Problems in Information Transmission, 1, 3-11.

Lempel, A. and Ziv, J. (1976). "On the Complexity of Finite Sequences." *IEEE Transactions on Information Theory*, 22, 75.

Li, M. and Vityáni, P.M.B. (1991). "Combinatorics and Kolmogorov Complexity." In *Proceedings of the Sixth Annual IEEE Conference on Structure in Complexity Theory*.

Lloyd, S. and Pagels, H. (1988). "Complexity as Thermodynamic Depth." Annals of Physics, 188, 186-213.

Morowitz, H. J. (1955). "Some Order-Disorder Considerations in Living Systems." Bulletin of Mathematical Biophysics, 17, 81-86.

Mowshowitz, A. (1968a). "Entropy and the Complexity of Graphs I: An Index of the Relative Complexity of a Graph." Bulletin of Mathematical Biophysics, 30, 175-204.

Mowshowitz, A. (1968b). "Entropy and the Complexity of Graphs II: The Information Content of Graphs and Digraphs." Bulletin of Mathematical Biophysics, 30, 175-204.

Mowshowitz, A. (1968c). "Entropy and the Complexity of Graphs III: Graphs With Prescribed Information Content." Bulletin of Mathematical Biophysics, 30, 175-204.

Rashevsky, N. (1955). "Life, Information Theory, and Topology." Bulletin of Mathematical Biophysics, 17, 229-235.

Rényi, A. (1961). "On Measures of Entropy and Information." In *Proceedings* of the Fourth Berkeley Symposium on Mathematical Statistics and Probability. Berkeley: University of California Press.

Roberts, F. (1969). "On the Boxicity and Cubicity of a Graph." In W.T. Tutte (Ed), Recent Progress in Combinatorics. New York: Academic Press.

Shannon, C. (1948). "A Mathematical Theory of Communication." Bell System Technical Journal, 27, 379-423.

Trucco, E. (1956). "On the Information Content of Graphs: Compound Symbols; Different States for Each Point." Bulletin of Mathematical Biophysics, 18, 237-253.

Wolpert, D. H. and Macready, W. G. (1998). "Self-Dissimilarity: An Empirical Measure of Complexity." Santa Fe Institute Working Paper.