Techniques Used to Estimate Limit Velocity in Ballistics Testing with Small Sample Size.

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ABSTRACT:

The US Army Research Laboratory (ARL) is currently conducts tests on anti-ballistic armor for military uses. This research is concerned with determining the limit velocity (v_L) of different target penetrator combinations. The limit velocity is the highest velocity a penetrator can have without penetrating the targe. Unfortunately, penetration processes are highly complex and an effective first principles derivation of v_L has not been discovered. Estimation of v_L is therefore done empirically. Furthermore, ballistics tests can be very expensive, resulting in a small size sample with which to perform statistical data analysis.

There are two ballistics testing methods commonly used to estimate v_L. The Jonas-Lambert method involves measuring the residual velocity of the projectile after perforation. The bisection method or V_{50} simply evaluates the perforation without residual velocity. The second method is significantly less expensive.

Simulation is used to model both of the common ballistics testing methods as well as several new approaches to ballistics testing. The results are evaluated and compared for statistical significance and accuracy. This work suggests that the bisection method is more accurate when sample size is small. This discovery could provide considerable cost savings to ballistics testing at ARL.

KEY WORDS: limit velocity, striking velocity, residual velocity, long-rod penetrators, ballistic testing, least squares, regression, small sample size

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INTRODUCTION

Problem

The US Army Research Laboratory, ARL, is currently responsible for testing the armor the military uses. This research is concerned with determining the limit velocity, V_L, for different types of armor. The limit velocity is the fastest velocity a round being shot at a certain type of armor can have without penetrating that armor. Knowledge of the limit velocity will provide a metric for the optimality of a given type of armor. In addition, the limit velocity can be used to establish the maximum velocity an enemy projectile can have without allowing a penetration.

Unfortunately, there have been no effective first principle derivations for limit velocity developed, therefore empirical testing is used to find an accurate prediction. However, this experimental testing is extremely expensive due to the number of real rounds fired, the armor that is used for the testing, and any equipment used for measuring data. Therefore, a method must be developed to minimize the sample size of shots fired while still allowing a true prediction for limit velocity.

Currently there are two techniques for this testing: The Jonas – Lambert Method which uses the speed of the round both before and after penetration and Bisection Method, or V_{50}, which does not. The latter is much less expensive. This study will compare the current methods and conclude which one is optimal in terms of accuracy, cost, and statistical significance. Additionally, different techniques that are not being used will be developed. These new methods will also be compared with the two current techniques to determine if a "better" system for testing exists.

Background Information

The limit velocity is defined as the highest striking velocity a piece of armor can withstand without allowing a complete penetration by a ballistic round. Thus, each piece of armor will have a different limit velocity based upon the type of round fired at it. This relationship is seen in Figure 1 below:

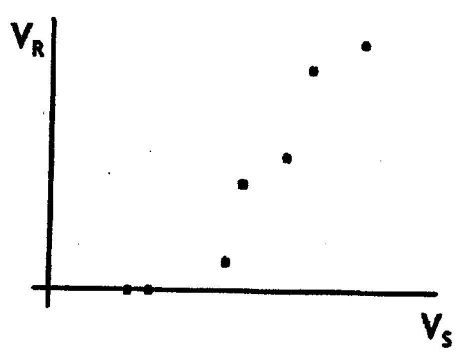

Figure 1: V_S v V_R

V_R represents the residual velocity, the speed of the round after penetration occurs, and V_S is the striking velocity, the speed of the round before contact with the armor. Every striking velocity that has a positive residual velocity represents a complete penetration. As the residual velocity becomes smaller, the striking velocity approaches the limit velocity. The limit velocity is the fastest striking velocity that has a residual velocity of zero.

Finding the limit velocity in a perfect world with the above information would be fairly simple. All that would be required is a function that describes the relationship between the striking velocity and residual velocity. The root of this function could be solved for and this would be the limit velocity. The impact dynamics in this type of testing, however, do not act in a perfect way. An extensive search of the literature reveals that an accurate relationship between the striking velocity and the residual velocity has not yet been derived. Figure 2 shows a scatter plot of a typical striking velocity versus residual velocity relationship. In the real world there is a significant amount of error involved and there exists a zone of mixed results.

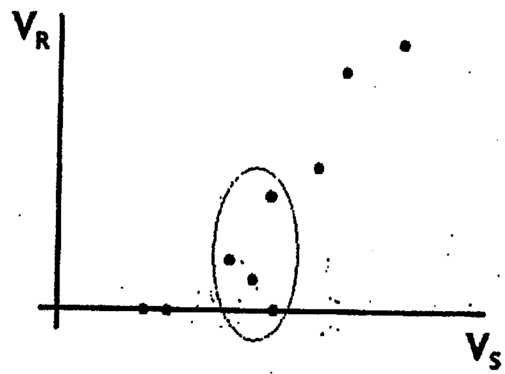

Figure 2: Zone of Mixed Results

The circled region in Figure 2 is referred to as a "zone of mixed results." As a result of unknown factors, some test shots fired at lower striking velocities can produce larger residual velocities. This makes finding a relationship between the residual and striking velocities very difficult. The armor is not flawless and thus some striking velocities that are above the limit velocity will not completely penetrate the armor. Likewise, some striking velocities below the limit velocity will completely penetrate the armor. This zone is real and will not disappear. Every test researched contained this error to some extent. For the purposes of this research, this zone is assumed to be small enough to not greatly affect the results and will not be included.

METHODS

Bisection, or V_{50}, Method

There are two common methods for testing limit velocity. Neither technique has been proven "optimal" yet. The cheaper of the two methods is The Bisection, or V_{50}, Method. Based on the Intermediate Value Theorem, this technique looks for the limit velocity by treating the V_S v V_R plot as a continuous and differentiable function, f. The function is defined by an interval $[a,b]$ with $f(a)$ and $f(b)$ of opposite signs. However, for this curve, zero values are considered to have negative values because there are no negative residual velocity values. By the Intermediate

Value Theorem, there must exist a point, p, in $[a,b]$ where $f(p)=0$. This p represents the limit velocity. When $f(p)=0$, the striking velocity will equal the largest value that has a corresponding $V_R=0$ and thus $V_L = V_S$. The method repeatedly halves the subintervals of $[a,b]$ while maintaining p in the subinterval of interest. Eventually this will find p to within a reasonable degree of accuracy.

When testing armor, this method starts with determining the brackets that the limit velocity is thought to exist in. A good ballistician is assumed to be able to predict brackets to within +/- 50 m/s. This bracket was chosen based on the recommendation of Dr deRosset, an ARL scientist who is active in this type of testing. Each round is then shot and if a complete penetration occurs, the lower half of the bracket is halved and a new shot is taken at this lower midpoint value. If a complete penetration does not occur, the upper half of the bracket is halved and the next shot is taken at this higher midpoint value. This technique is seen in Figure 3 below:

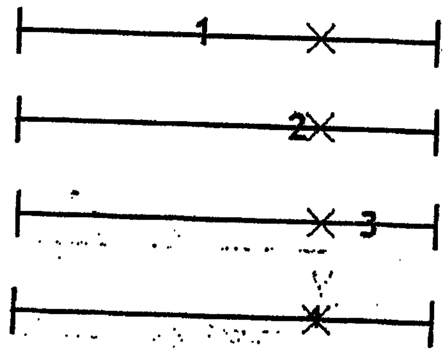

Figure 3: The Bisection Method

The bracketed area is first halved with the shot labeled "1." The X shows the actual limit velocity. The first shot will not have a complete penetration because the 1 is to the left, is lower, of the limit velocity. The second shot, labeled "2" will then halve the upper half of the original bracket. This iterative method will continue until an answer of desired precision is determined.

This method is relatively cheap because of the required data and equipment. This test is only concerned with whether or not a penetration occurs. Therefore, only the price for each shot and the pieces of armor used cover the price of the entire test.

The Jonas - Lambert Method

The Jonas-Lambert Method is more expensive than the Bisection Method. This test requires the measurements of the residual and striking velocities as opposed to only being concerned with whether a complete penetration occurs. This technique tries to find a relationship between the striking velocity and the residual velocity. If a function is found that predicts the residual velocity from the striking velocity, the root of this function will be the limit velocity. A graphic representation of this is in Figure 4 below:

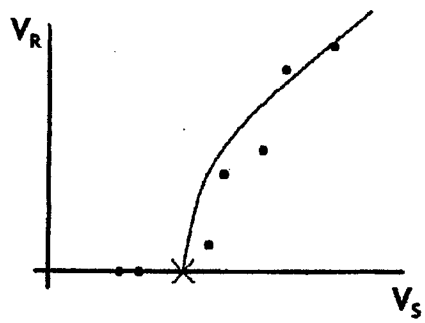

Figure 4: The Jonas-Lambert Curve

It can be seen in Figure 4 above that the root of the function is equal to the limit velocity. The equation of this line is of the form:

$$V_R = \begin{cases} \alpha(V_S{}^p - V_L{}^p)^{1/p} + \varepsilon, V_L < V_S \\ 0, 0 \le V_S \le V_L \end{cases}$$

This equation is derived from The Law of Conservation of Energy. The derivation and an explanation of it is included in Appendix A.

This method requires at least three shots where the striking velocity is greater than the limit velocity to effectively estimate the limit velocity. Three shots are required in order to give three separate equations which in turn allow for solving the three unknown parameters, excluding the error term. After defining The Jonas – Lambert Curve, the limit velocity can be found by simply finding its root.

An efficient application of The Jonas – Lambert Method involves taking three shots that completely penetrate. These three shots will provide the necessary data to solve for all the unknown parameters except the error term. The calculated limited velocity is then used as the next shot's striking velocity. If a complete penetration occurs on this next shot, the variables are re-solved and the next shot will be at the latest calculated value of the limit velocity. If a complete penetration does not occur, re-shoot at a higher velocity until a penetration occurs. Once a penetrations occurs, continue with the method. Shots will continue to be fired until an answer of desired accuracy is found.

V_S v V_R Relationships

The first method is looking for a relationship, similar to the one used in the Jonas – Lambert Method, between the residual and striking velocities. If an accurate function which describes the residual velocity based on the striking velocity can be found, the root of this function will provide a more accurate estimate of the limit velocity.

This method requires the same testing techniques and data and thus will be similar in price to The Jonas – Lambert Method. The difference is that common relationships are used to describe the data rather than some complex equation. This method looks at linear, exponential, power and logarithmic relationships.

Golden Ratio Method

The Golden Ratio Method is an extension of The Bisection Method and is very similar. This method uses *The Golden Ratio,* a ratio that appears consistently throughout nature, as the way of determining the bracket used rather than simply using a halving technique. Like the Bisection Method, this method is only concerned with whether or not the round penetrates the piece of armor or not. Therefore, the price of this method only deals with how many shots are taken. Additionally, the same initial bracket of +/- 50 m/s will be used to remain consistent with the evaluation of The Bisection Method.

The algorithm for this method starts with an initial bracket of +/- 50 m/s and shoots at the middle. The initial bracket is divided into an upper and a lower 61.8 % (the golden ratio) portion. If the first shot penetrates, the lower portion is used as the second bracket and if a penetration does not occur, the upper portion is used. The second shot then shoots at the middle of this second bracket. Once again, if a penetration occurs, the lower 61.8% of the second bracket is used as the third bracket, and if no penetration occurs, the upper 61.8% portion is used. This iterative technique continues until a limit velocity of desired accuracy is determined. This process is seen in Figure 5.

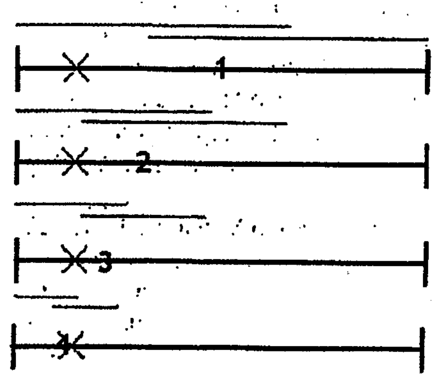

Figure 5: The Golden Ratio Method

The lower of the two overlapping lines represents the upper 61.8% of the bracket and the higher of the two is the bottom 61.8%. The first shot, labeled "1," bisects the initial bracket in half. The "X" represents the limit velocity and thus, the first shot is a penetration. The lower portion is then used as the next bracket. The "2" marks the second shot which halves the bottom 61.8% of the bracket. This shot is a penetration as well and thus the lower portion is used for the third bracket. The second green line down becomes the new bracket and the third shot will then half this new bracket. This iterative method continues until the limit velocity is found to within a desired level of accuracy.

Residual Energy v The Angle of the Projectile Relationships

This method deals with trying to find a relationship between the angle of the projectile, with respect to the armor, when it strikes the target and the residual energy. The residual energy is the energy of the projectile after it penetrates the armor. As the round hits the armor, energy is absorbed by the armor, the round breaks up, and energy is contained in all of the round fragments.

Using the principles of conservation of energy, an equation is derived that gives the limit velocity as a function of this residual energy and the striking mass of the round. The equation is of the form:

$$V_L = \sqrt{\frac{2E}{m_S}}$$

The derivation of this is in Appendix B. Therefore, if a relationship can be found which describes this residual energy, E, based on the angle, then the limit velocity can be found.

This method investigated may be costly because data must be gathered on the striking angle of the round. The particular factors of the angle in this method are the pitch, yaw and resultant of the projectile as it strikes the armor.

Residual Energy v V_R Relationships

This method investigates a relationship between the residual energy and the residual energy. The goal is to find a function that describe the residual energy based on the residual velocity. The root of this function will yield the residual energy at the limit velocity, which can then be used to solve for the limit velocity.

This method requires some costly data collection as well. The residual and striking velocity as well as the striking and residual mass of the round must be measured to define the function. The residual velocity is obviously needed because it is the independent variable. The other pieces of data are needed to solve for the residual energy. The residual energy is found using the equation of the form:

$$\frac{1}{2}M_S V_S^2 - \frac{1}{2}M_R V_R^2 = \sum E$$

The derivation of this equation is located in Appendix B.

RESULTS

Bisection or V_{50} Method

The Bisection Method can produce a limit velocity to any desired accuracy. The algorithm increases accuracy with every iteration performed. To test this method, a code in Microsoft Excel was developed that takes successive shots until the desired accuracy is produced. For the test, sample limit velocities were used from data given from ARL. A few examples of this code is in Appendix C. Table 1 shows the results of all the tests conducted.

trial	shots for a tolerance of 5 m/s	shots for a tolerance of 1 m/s
1	3	5
2	3	6
3	4	5
4	3	6
5	2	5
6	3	6
7	2	6
8	3	3
9	2	6
10	3	6
11	3	6
12	4	6
average shots	2.916666667	5.5

Table 1: Bisection Results

It can be seen in Table 1 that this method produces a result within 5 m/s of the true limit velocity with 2.9 shots on average. A point of interest with these results is the consistency. There are no outliers in the data. The most shots ever needed to reach this accuracy of 5 m/s is only 4 shots.

The Jonas - Lambert Method

The Jonas – Lambert Method does not accurately and completely describe the behavior of a projectile penetrating a piece of armor. For example, two shots that have the exact same striking velocity may result in different residual velocities. This is the main reason why this method has problems. To test this method, data sets obtained from ARL, which include all the information needed to use The Jonas – Lambert Method, were used. This data is in Appendix D. The Jonas – Lambert Method's estimate was compared with the actual limit velocity. The results are shown in Table 2.

complete penetrations	shots	a	p	V_L estimate (m/s)	V_L act (m/s)	error (m/s)
4	6	0.89	2.59	1365.684	1373	7.316
3	5	0.9	2.49	1082.559	1088	5.441
3	5	0.86	2.39	1303.539	1324	20.461
3	8	1	2.99	1136.125	1164	27.875
3	9	0.88	2.59	1178.759	1239	60.241
2	9	0.97	5.69	1341.986	1355	13.014
average	7					22.39133

Table 2: Jonas – Lambert Results

With the data supplied by ARL, The Jonas – Lambert Method was used to estimate the limit velocity for each shot. To arrive at the limit velocity estimation seen in Table 2, every shot from each data set is used to calculate a limit velocity and then the average from the entire data set gives the final estimate. A table showing this data, along with a sample algorithm for this method, is in Appendix E.

The data shows that this method on average takes 7 shots to reach a limit velocity that is accurate to within 22.4 m/s. However, this data can be deceiving. First, this average error, is just that, an average. Table 2 shows that outliers existed where the method was only accurate to within 60 m/s. The average number of shots is a better representation of the data, because the most shots ever taken in the testing was 9. However, there is no substantial relationship between the number of shots taken and then accuracy of the estimate. For example, 5 shots produced an estimate accurate to within 5.4 m/s and 9 shots gave two different less accurate estimates: 60.2 m/s and 13.0 m/s. This is different from what is expected; in theory, as the number of shots increase, the accuracy should increase as well.

V_S v V_R Relationships

This method was stopped before any results were calculated because the findings were so inaccurate. The problem is that the V_S v V_R data does not fit into a "normal" (logarithmic, exponential, power, or linear) function. Out of these functions, the best relationship was the logarithmic. The results for this method are shown in Table 3.

trial	shots	penetrations	V_L actual (m/s)	V_L estimate (m/s)	error (m/s)
1	6	4	1366	1343.46	22.54043
2	5	3	1083	1074.171	291.8289
3	5	3	1304	1247.197	118.8031
	5.333333				144.3908

Table 3: V_S v V_R Results

Only three sequences of this method occurred because the results were already unacceptable. As you can see the average error was already with 144.4 m/s and one estimate was off by almost 300 m/s. This is unacceptable because ballisticians can usually guess to within 100 m/s. Sample calculations for this method are in Appendix F.

Golden Ratio Method

This method was tested in the same manner as The Bisection Method. Microsoft Excel was used to perform iterations of the algorithm until a desired estimate was calculated. A few sequences of the test is in Appendix G. The results of this method are shown in Table 4.

trial	shots for a tolerance of 5 m/s	shots for a tolerance of 1 m/s
1	2	6
2	3	5
3	5	6
4	4	4
5	3	5
6	3	3
7	2	4
8	4	4
9	3	6
10	4	4
11	3	5
12	3	5
average shots	3.25	4.75

Table 4: Golden Ratio Results

The results from Table 4 show that this method takes 3.3 shots on average to achieve an estimate within 5 m/s. It should be noticed that, like the bisection method, this method is very consistent and has no outliers. A point of interest about this method in comparison to the Bisection method is the difference between achieving an estimate to within 1 m/s versus 5 m/s. It takes more shots for The Golden Ratio Method than the Bisection Method to achieve an estimate to within 5 m/s but less shots for 1 m/s.

Residual Energy v The Angle of the Projectile Relationships

No estimates of the limit velocity are presented because no significant relationships could be found between the residual energy and the angle of the projectile. Out of the three factors, the yaw of the round gave the best relationship but it was still very insignificant and inconsistent. The data supplied by ARL along with the calculated residual energy is located in Appendix H. Additionally, Appendix I contains a few sample regressions between beta and the residual energy. As mentioned earlier, yaw produced the most significant results, therefore only these results are shown because the other attempts proved fruitless. The exponential trend line is displayed because it was the only function that could be possibly used for each data set.

Residual Energy v V_R Relationships

The first step for this method is to find a relationship between the residual energy and the residual velocity. For 5 data sets, a relationship of enough significance was found to continue with the test. Appendix J shows a few of the sample regressions between these two variables as well as sample calculations. Table 5 shows the results of this method.

trial	complete penetrations	shots	relationship significance (r squared)	V_L estimate (m/s)	V_L actual (m/s)	error (m/s)
1	4	6	0.693	1345.095	1373	27.905
2	3	5	0.729	1076.384	1088	11.616
3	3	5	0.983	1245.647	1324	78.353
4	3	8	0.83	995.6095	1164	168.3905
5	4	9	0.252	978.7987	1239	260.2013
		6.6				109.2932

Table 5: Residual Energy v V_R Relationships Results

This method produces very fickle results. Some tests show that an error of only 11.6 m/s can be reached by only taking 5 shots, but others result with a 260.2 m/s error with 9 shots. The average results show this method can produce an estimate with an error of 109.3 m/s with 6.6 shots. This test is automatically discarded because the error is larger than the bracket that the ballisticians can initially guess.

CONCLUSION

The Bisection Method is the optimal method to find the limit velocity. This method is considered the best method based on cost, accuracy, and reliability. The Bisection Method is the cheapest way to find the limit velocity. Not only does this test not require any expensive data collection equipment but it requires the least amount of shots to run the test. More importantly, this method provided the most accurate results. For a desired accuracy of 5 m/s, The Bisection Method is able to achieve these results with every test. Lastly, this method is the most reliable as well. Outliers do not exist in this method; plus or minus two shots, this method always returns an accurate answer.

The Golden Ratio Method places a close second to The Bisection Method for the same three reasons discussed above. However, The Bisection Method is able to outperform the Golden Ratio Method for the number of shots to produce an estimate within 5 m/s and thus The Bisection Method is chosen as the better of the two.

RECOMMENDATION

The following algorithm should be applied for finding the limit velocity.

1. -Estimate the limit velocity and shoot at this velocity.

2. -If a complete penetration occurs, use this velocity as the right limit.
-If a complete penetration does not occur, use this velocity as the left limit.

3. -Shoot the next shot at + or - 75 m/s of the left or right limit depending on the result of the first shot.

4. -If the opposite result for the second shot occurs compared with the first shot, proceed with the bisection algorithm.
-If the same result for the second shot occurs as compared with the first shot, take the next shot at + or - 25 m/s of the last shot in order to ensure a + or – 100 m/s bracket is achieved. If the opposite result from the previous shots still does not occur, continue taking shots at + or – 25 m/s until a complete penetration or no penetration occurs, depending on what result is needed. Once a bracket of a complete penetration and no penetration is developed, proceed with the bisection algorithm.

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Leonard, Wendy. *The Effect of Nose Shape on Depleted Uranium (DU) Long – Rod Penetrators(ARL – TR – 1505).* Aberdeen Proving Grounds, MD: Army Research Laboratory, 1997.

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Appendix A

The Jonas - Lambert Derivation

Start with the conservation of energy equation with an error term describing the energy lost in the armor and fragments of the round penetrating through the armor.

$$\frac{1}{2}m_S V_S^2 = \frac{1}{2}m_R V_R^2 + E$$

The variable E represents the energy at the limit velocity. Therefore E is equal to the kinetic energy of the round when the limit velocity occurs.

$$E = \frac{1}{2}m_L V_L^2$$

The Jonas – Lambert Theorem assumes that the mass of the round does not change at all points in the test. Therefore mass can be eliminated from the equation.

$$V_S^2 = V_R^2 + V_L^2$$

$$\therefore V_R = (V_S^2 + V_L^2)^{\frac{1}{2}} + \varepsilon$$

There is an error term added because some of the assumptions used for the cancellation of the masses are not exact. Jonas – Lambert then changes the 2's to p's and adds an alpha term to further help describe any error. The final form is:

$$V_R = \alpha(V_S^p - V_L^p)^{1/p} + \varepsilon$$

Appendix B

Residual Energy Derivation

Start with the conservation of energy equation. The energy of the striking round, the left side of the equation, will equal the energy of the residual round plus the energy lost in the armor and with fragments of the round, the right side of the equation.

$$\frac{1}{2}m_S V_S^2 = \sum \frac{1}{2} m_R V_R^2 + E$$

The energy of the main piece of the residual round is then subtracted from both sides. There is no summation attached to the kinetic energy of the residual round because the term only deals with the main residual mass that penetrates the armor. Now the term on the right side of the equation represents all of the energy that is lost into the armor and through the kinetic energy in round fragments.

$$\frac{1}{2}m_S V_S^2 - \frac{1}{2} m_R V_R^2 = \sum E$$

At the limit velocity, the residual velocity of the round will be 0. The equation is changed to this instance. The residual velocity goes to zero and the striking velocity becomes the limit velocity.

$$\frac{1}{2}m_S V_L^2 = \sum E$$

This equation is manipulated to give the limit velocity as a function of the energy lost and the mass of the striking round.

$$V_L = \sqrt{\frac{2E}{m_S}}$$

Appendix C

Sample Bisection Method Tests

		left limit	1300				Tolerance	5
V_L actual	1366	right limit	1400					
				left	right		Success?	
shot 1	1350			1300	1400		Fail	
shot 2	1375			1350	1400		Fail	
shot 3	1362.5			1350	1375		Success	

Trial 1

		left limit	1000				Tolerance	5
V_L actual	1083	right limit	1100					
				left	right		Success?	
shot 1	1050			1000	1100		Fail	
shot 2	1075			1050	1100		Fail	
shot 3	1087.5			1075	1100		Success	

Trial 2

		left limit	1300				Tolerance	5
V_L actual	1304	right limit	1400					
				left	right		Success?	
shot 1	1350			1300	1400		Fail	
shot 2	1325			1300	1350		Fail	
shot 3	1312.5			1300	1325		Fail	
shot 4	1306.25			1300	1312.5		Success	

Trial 3

		left limit	1100				Tolerance	5
V_L actual	1136	right limit	1200					
				left	right		Success?	
shot 1	1150			1100	1200		Fail	
shot 2	1125			1100	1150		Fail	
shot 3	1137.5			1125	1150		Success	

Trial 4

note – all velocities are measured in m/s.

Appendix D

Supplied Data

V_R	V_S	a	p	V_L actual
293	1405	0.89	2.59	1373
360	1383			
196	1378			
647	1437			
360	1123	0.9	2.49	1088
359	1108			
91	1089			
318	1330	0.86	2.39	1324
383	1344			
473	1371			
824	1285	1	2.99	1164
772	1232			
536	1168			
618	1254	0.88	2.59	1239
419	1288			
561	1265			
801	1361			
1046	1384	0.97	5.69	1355
655	1370			

note – all velocities are measured in m/s.

Appendix E

Jonas – Lambert Calculations

V_R	V_S	a	p	V_L estimate	Error	Average	V_L actual
293	1405	0.89	2.59	1392.25619	26.57185	1365.684	1373
360	1383			1360.596933	-5.08741		
196	1378			1373.381776	7.697431		
647	1437			1336.50248	-29.1819		
360	1123	0.9	2.49	1087.672754	5.11327	1082.559	1088
359	1108			1072.183109	-10.3764		
91	1089			1087.822589	5.263105		
318	1330	0.86	2.39	1303.525227	-0.01379	1303.539	1324
383	1344			1302.999391	-0.53963		
473	1371			1304.092447	0.553425		
824	1285	1	2.99	1159.344667	23.21946	1136.125	1164
772	1232			1120.382441	-15.7428		
536	1168			1128.648505	-7.4767		
618	1254	0.88	2.59	1137.727821	-41.0315	1178.759	1239
419	1288			1249.301445	70.54216		
581	1265			1177.459531	-1.29975		
801	1361			1150.54833	-28.211		
1046	1384	0.97	5.69	1318.302238	-23.6835	1341.986	1355
655	1370			1365.669192	23.68348		

Data

	x	y	estimate	sq error	SSE		estimate			actual	
shot	V_S	V_R	V_R								
1	1405	416.3047506	446.899343	936.0291	16806.04		a	0.814753		a	0.89
2	1383	264.3929186	325.152452	3691.721			p	2.780557		p	2.59
3	1375	141.7757196	252.130991	12178.29			V_L	1367.14		V_L	1373
4											
5											
6											

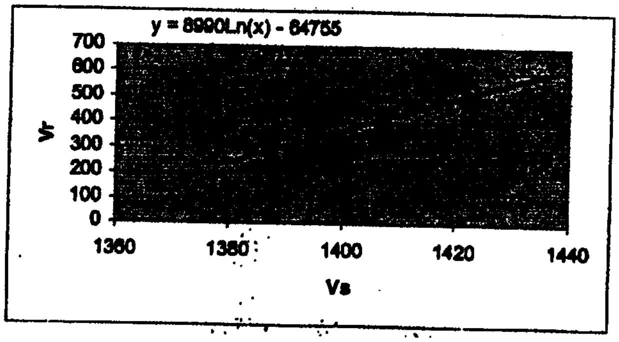
Sample Algorithm

note – all velocities are measured in m/s.

Appendix F

V_S v V_R Calculations

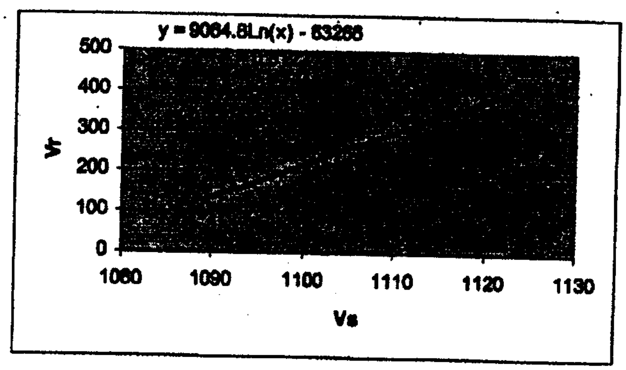
	V_S	V_R	V_L actual
shot 1	1405	293	1366
shot 2	1383	360	
shot 3	1378	196	
shot 4	1437	647	



V_L estimate	1343.46

Trial 1

	V_S	V_R	V_L actual
shot 1	1123	360	1083
shot 2	1108	359	
shot 3	1089	91	



V_L estimate	1074.171

Trial 2

note – all velocities are measured in m/s.

Appendix G

Sample Golden Ratio Method Tests

			x lo	1300				
V_L actual	1366		x hi	1400			Tolerance	5
T	x lo	x1	x2	x hi	shot	Approximation		Success?
0	1300	1338.2	1361.8	1400	1	1350		Fail
1	1338.2	1361.8	1376.392	1400	2	1369.1		Success

Trial 1

			x lo	1000				
V_L actual	1083		x hi	1100			Tolerance	5
t	x lo	x1	x2	x hi	shot	Approximation		Success?
0	1000	1038.2	1061.8	1100	1	1050		Fail
1	1038.2	1061.8	1076.392	1100	2	1069.1		Fail
2	1061.8	1076.392	1085.408	1100	3	1080.9		Success

Trial 2

			x lo	1300				
V_L actual	1304		x hi	1400			Tolerance	5
t	x lo	x1	x2	x hi	shot	Approximation		Success?
0	1300	1338.2	1361.8	1400	1	1350		Fail
1	1300	1323.608	1338.2	1361.8	2	1330.9		Fail
2	1300	1314.592	1323.608	1338.2	3	1319.1		Fail
3	1300	1309.018	1314.592	1323.608	4	1311.8038		Fail
4	1300	1305.574	1309.018	1314.592	5	1307.2962		Success

Trial 3

			x lo	1100				
V_L actual	1136		x hi	1200			Tolerance	5
t	x lo	x1	x2	x hi	shot	Approximation		Success?
0	1100	1138.2	1161.8	1200	1	1150		Fail
1	1100	1123.608	1138.2	1161.8	2	1130.9		Fail
2	1123.608	1138.2	1147.211	1161.8	3	1142.7038		Fail
3	1123.608	1132.624	1138.2	1147.211	4	1135.409052		Success

Trial 4

note – all velocities are measured in m/s.

Appendix H

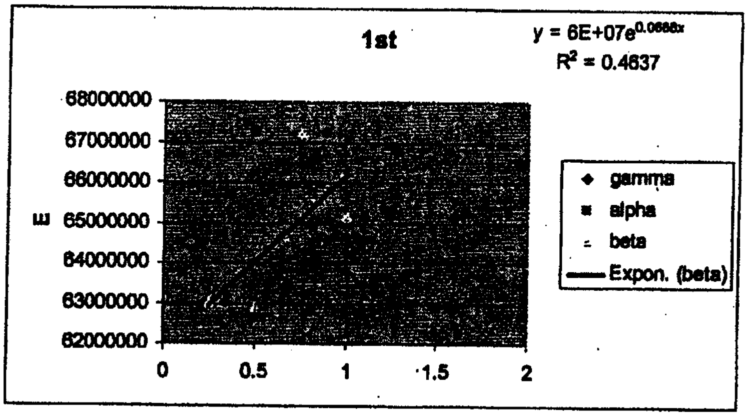
Data For The Residual Energy v The Angle of the Projectile Relationships

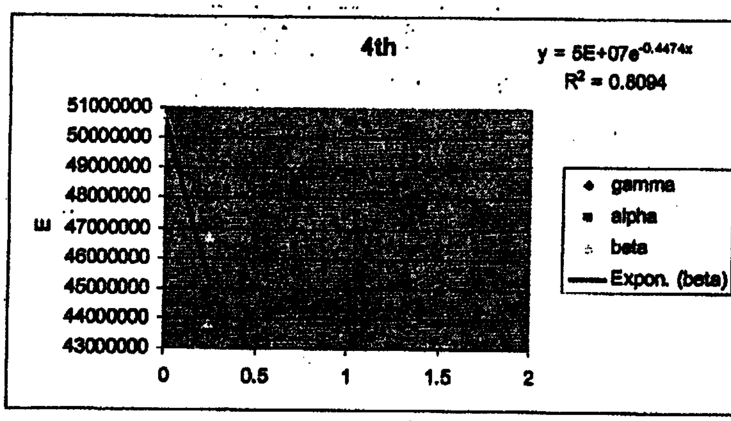
m_S	V_S	m_R	V_R	E	gamma	alpha	beta	V_L actual
66.33	1378	4.37	196	62892549	1.58	1.5	0.5	1373
66.32	1383	6.38	360	63011343	0.34	0.25	0.25	
66.25	1405	5.54	293	65151776	1.6	1.25	1	
66.38	1437	6.21	647	67236540	0.89	0.5	0.75	
66.2	1123	9.26	360	41143322	0.79	0.25	0.75	1088
66.29	1108	7.89	359	40182488	0.56	0.25	0.5	
66.37	1089	6.91	91	39326178	0.34	0.25	0.25	
65.78	1330	6.55	318	57847940	0.25	0	0.25	1324
65.79	1344	7.56	383	58864938	0.56	0.25	0.5	
65.88	1371	5.73	473	61274391	0.7	0.5	0.5	
65.75	1285	10.9	824	50583603	1	1	0	1164
65.8	1232	10.96	772	46670417	0.56	0.5	0.25	
65.82	1168	7.39	536	43835053	1.51	1.5	0.25	
66.22	1254	10.45	618	50070452	0.56	0.25	0.5	1239
66.05	1288	7.24	419	54151095	1.58	1.5	0.5	
66.03	1265	9.09	561	51401021	0.75	0.75	0	
65.97	1361	7.74	801	58615812	0.89	0.75	0.5	

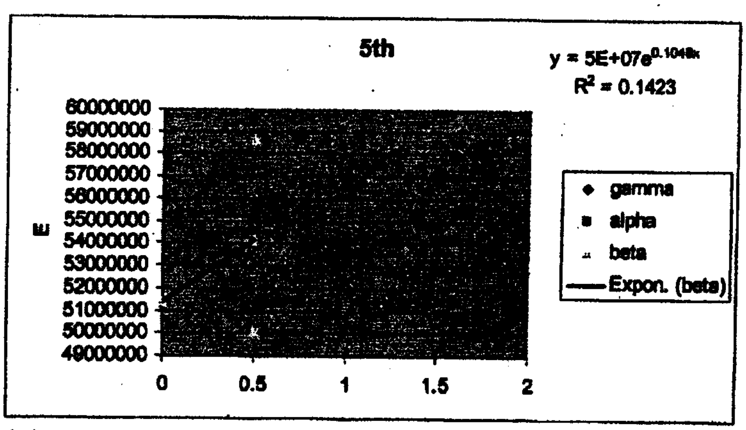
note – all velocities are measured in m/s.

Appendix I

Sample Regressions For The Residual Energy v The Angle of the Projectile Relationships



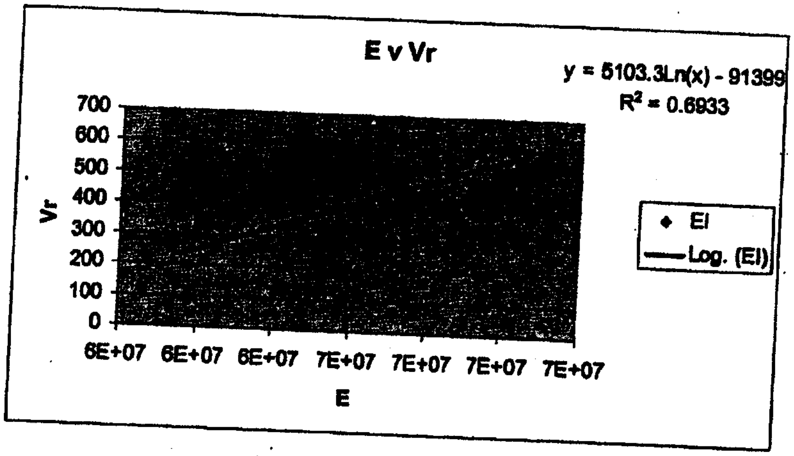




note – gamma, alpha, and beta represent the resultant angle, pitch and yaw of the round, respectively

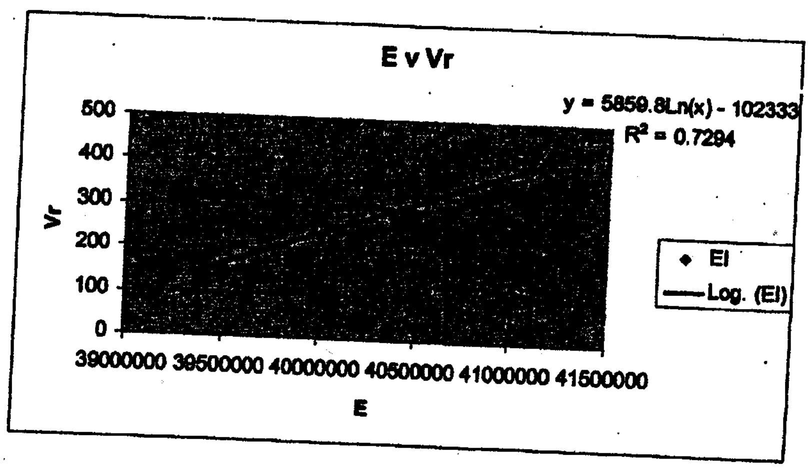
Appendix J

Sample Regressions For The Residual Energy v V_R



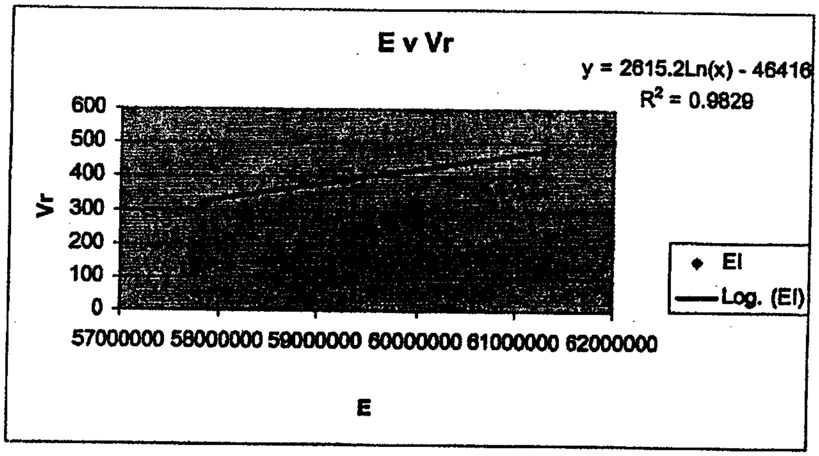
m_s	E at V_L estimate	V_L estimates	V_L actual	V_L estimate	error
66.33	59995724.86	1344.993395	1373	1345.095	27.90496
66.32	59995724.86	1345.094793			
66.25	59995724.86	1345.805221			
66.38	59995724.86	1344.486749			

Trial 1



m_s	E at V_L estimate	V_L estimates	V_L actual	V_L estimate	error
66.2	38399896.01	1077.087603	1088	1076.384	11.6163
66.29	38399896.01	1076.356189			
66.37	38399896.01	1075.707293			

Trial 2



m_s	E at V_L estimate	V_L estimates	V_L actual	V_L estimate	error
65.78	51061786.44	1245.994329	1324	1245.647	78.35257
65.79	51061786.44	1245.899631			
65.88	51061786.44	1245.048315			

Trial 3

note – all velocities are measured in m/s.